2B

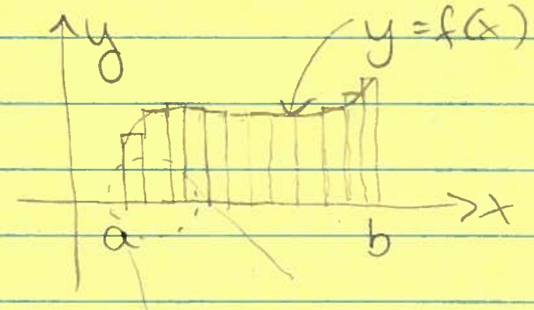
L9
§6.5

Average value of a function

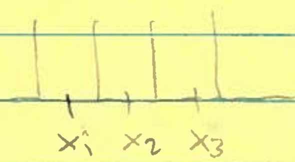
The average of 1, 3, −1, 5 is:

$$\frac{1+3-1+5}{4}. \qquad (*)$$

How can we generalize this notion from numbers to functions? One approach is to use Riemann rectangles:



rectangles = n



$$\frac{f(x_1)+f(x_2)+\cdots+f(x_n)}{n} \qquad (**)$$

Certainly (**) "looks like" (*) but how many "samples" of f(x) should we choose, i.e. how big should n be?

Well, if we want to define the average of a function (rather than just a finite # samples from the function), we should "send n to infinity" so that we sample all of the function:

Definition: $$f_{ave} = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} f(x_i)$$

We can simplify the RHS as follows.

Recall that the base of each rectangle is:

$$\Delta x = \frac{b-a}{n} \Rightarrow \frac{1}{n} = \frac{\Delta x}{b-a}.$$

Thus:

$$f_{ave} = \frac{1}{b-a}\lim_{n\to\infty}\sum_{i=1}^{n} f(x_i)\,\Delta x$$

i.e. $$\boxed{f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx} \qquad (1)$$

EX: Find the average of $f(x) = 2xe^{-x^2}$ on $[0,2]$.

Sol^{n}

$$f_{ave} = \frac{1}{2-0}\int_0^2 2xe^{-x^2}\,dx$$

$u = x^2$

$du = 2x\,dx$

$x=0 \Rightarrow u=0$

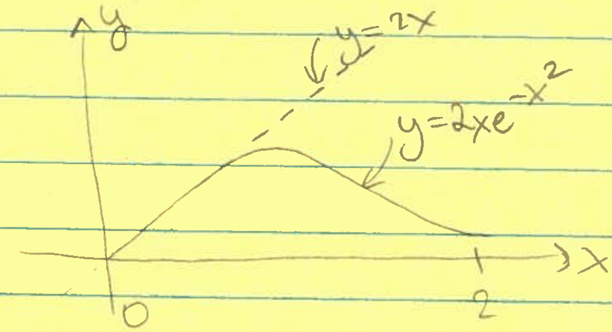
$x=2 \Rightarrow u=4$

$$= \frac{1}{2}\int_0^4 e^{-u}\,du$$

$$= \frac{1}{2}\left[-e^{-u}\right]_0^4 = \frac{1}{2}\left(1-e^{-4}\right)$$

<u>Ex</u> In previous example, find c s.t. $f_{ave} = f(c)$.

<u>Sol^n</u>



<u>Note 1</u>: $e^{-x^2} = 1 - x^2 + \cdots$ (Taylor)

$\Rightarrow 2xe^{-x^2} \simeq 2x - 2x^3 + \cdots$

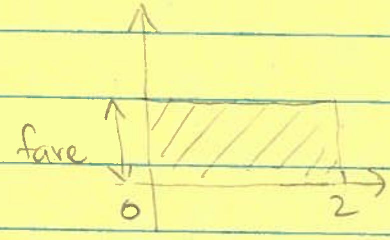
$\Rightarrow f(x) \simeq 2x \ (x \ll 1)$

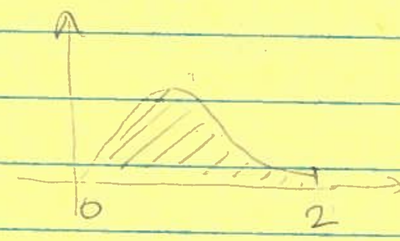
<u>Note 2</u>: e^{-x^2} goes to zero "faster" than $2x$ goes to ∞ as $x \to \infty$.

Now:

$$f_{ave}(b-a) = \int_a^b f(x)dx \quad \ldots \quad \text{cf.}(1).$$

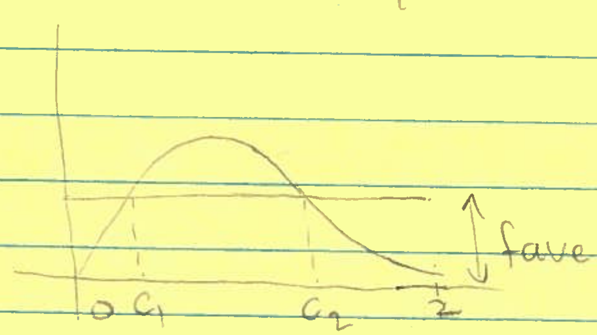
$\Rightarrow$





$a=0$; $b=2$

Thus fave must be smaller than maximum of $f(x)$ on $[0,2]$:



$\Rightarrow$ there are two places (c_1, c_2) where $f(c_1) = f(c_2) = f_{ave}$

In fact, the previous example illustrates a theorem:

<u>Mean-value Theorem for Integrals</u>:
If f is continuous on $[a,b]$, then $\exists$ a $c \in [a,b]$ s.t. $f(c) = f_{ave}$.

<u>EX</u> If f is continuous and $\int_1^3 f(x)\,dx = 8$, show that f takes on the value 4 at least once on $[1,3]$.

<u>Sol^{n}</u> $f_{ave} = \frac{1}{3-1}\int_1^3 f(x)\,dx = \frac{1}{2}\cdot 8 = 4$.

Thus, by MVT, $\exists$ c in $[1,3]$ s.t. $f(c) = 4$.

<u>EX</u> Find b s.t. $f_{ave} = 3$ on $[0,b]$ where $f(x) = 2 + 6x - 3x^2$.

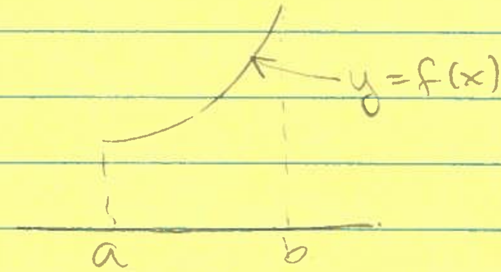
<u>Sol^{n}</u> $f_{ave} = 3$ on $[0,b] \Rightarrow 3 = \frac{1}{b-0}\int_0^b (2+6x-3x^2)\,dx$

$$= \frac{1}{b}\left[2x + 6\frac{x^2}{2} - 3\frac{x^3}{3}\right]_0^b = \frac{1}{b}\left[2b + 3b^2 - b^3\right]$$

i.e. $3 = 2 + 3b - b^2 \Rightarrow b^2 - 3b + 1 = 0$.

$$\Rightarrow b = \frac{3 \pm \sqrt{5}}{2} > 0$$

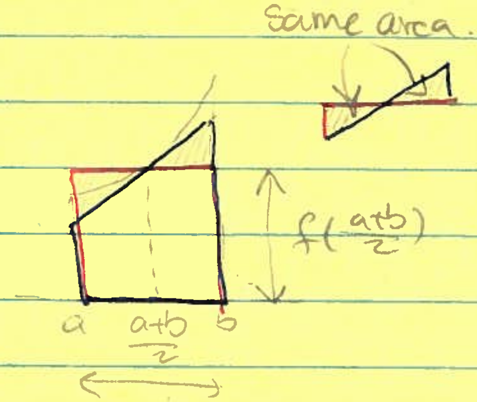
Ex Consider:



Show that $f_{ave} > f\left(\frac{a+b}{2}\right)$.

Sol^{n}

$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$



$> \frac{1}{b-a}$ area()

$= \frac{1}{b-a}$ area()

$= \frac{1}{b-a} \cdot f\left(\frac{a+b}{2}\right) \cdot (b-a)$

$= f\left(\frac{a+b}{2}\right)$.

Math 2B : Midterm # 1 Sample

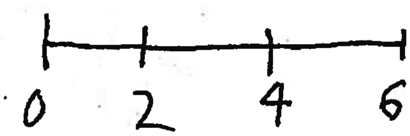
This exam consists of 5 questions. Problems # 1-3 are worth 15 points each and problems # 4 and 5 are worth 20 points each. There is a total of 85 available points. Read directions for each problem carefully. Please show all work needed to arrive at your solutions. Label all graphs. Clearly indicate your final answers.

1.) a.) Estimate the area under the graph of $f(x) = x^2 + x$ from $x = 0$ to $x = 3$ using 3 approximating rectangles and left endpoints. Width of each rectangle $= 1$.

$$1 \cdot f(0) + 1 \cdot f(1) + 1 \cdot f(2)$$

$$= 1 \cdot 0 + 1 \cdot 2 + 1 \cdot 6 = 8.$$

b.) Estimate the area under the graph of $f(x) = x - 1$ from $x = 0$ to $x = 6$ using 3 rectangles and midpoint approximation method. width $= 2$



$$2 \cdot f(1) + 2 \cdot f(3) + 2 \cdot f(5)$$

$$= 2 \cdot 0 + 2 \cdot 2 + 2 \cdot 4 = 12$$

c.) Find an expression for the area under the graph of $f(x) = x^2 + x$ from $x = 2$ to $x = 5$ as a limit of a Riemann sum. (You do not need to evaluate.)

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{5-2}{n} \cdot f(x_i^*)$$

We'll use right endpoints: $x_i^* = 2 + i \cdot \left(\frac{5-2}{n}\right)$

$$\lim_{n \to \infty} \sum_{i=1}^{n} \frac{3}{n} \cdot \left[\left(2 + i\frac{3}{n}\right)^2 + \left(2 + i\frac{3}{n}\right)\right]$$

2.) Evaluate each of the following indefinite integrals:

a.) $\int x\sqrt{3x^2-1}\,dx$

$u = 3x^2 - 1$ $\quad du = 6x\,dx$ $\quad \frac{1}{6}du = x\,dx$

$= \int \frac{1}{6}\sqrt{u}\,du = \frac{1}{6}\cdot\frac{2}{3}\cdot u^{\frac{3}{2}} + C$

$= \frac{1}{9}(3x^2-1)^{\frac{3}{2}} + C$

b.) $\int \frac{(1-\sin^2 x)}{\cos x}\,dx$ $= \int \frac{\cos^2 x}{\cos x}\,dx = \int \cos x\,dx = \sin x + C$

$\cos^2 x + \sin^2 x = 1$, so $\cos^2 x = 1 - \sin^2 x$

c.) $\int \sin(7\theta + 5)\,d\theta$

$u = 7\theta + 5$ $\quad \frac{1}{7}du = d\theta$

$\int \frac{1}{7}\sin(u)\,du = -\frac{1}{7}\cos(u) + C$

$= -\frac{1}{7}\cos(7\theta + 5) + C$

3.) a.) Find the average value of the function $f(x) = tan^3x \; sec^2x$ on the interval $\left[0, \frac{\pi}{4}\right]$.

$$\text{average} = \frac{1}{\frac{\pi}{4} - 0} \int_0^{\frac{\pi}{4}} \tan^3 x \underbrace{\sec^2 x}_{1/\cos^2 x} \, dx$$

$u = \tan x \qquad du = \sec^2 x \, dx$ (cf. p2 LS)

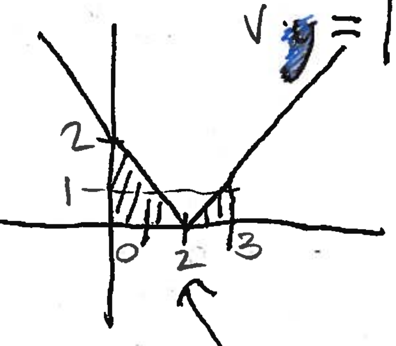
$$= \frac{4}{\pi} \int_{\tan 0}^{\tan \frac{\pi}{4}} u^3 \, du = \frac{4}{\pi} \int_0^1 u^3 \, du = \frac{4}{\pi} \left. \frac{u^4}{4} \right|_0^1$$

$$= \frac{1}{\pi}$$

b.) A particle moves along a line so that its velocity at time t is $v(t) = |2 - t|$. Find the displacement of the particle during the time period $0 \le t \le 3$.

$$\text{Displacement} = \int_0^3 |2 - t| \, dt$$

$v = |2 - t|$.



$$= \text{Area} = \frac{1}{2} \cdot 2 \cdot 2 + \frac{1}{2} \cdot 1 \cdot 1$$

$$= \frac{5}{2}$$

Alternative Sol^n

$$|2-t| = \begin{cases} 2-t & t < 2 \\ -(2-t) & t > 2 \end{cases}$$

$$\int^3 |2-t| dt$$

$$\int_0^2 (2-t)dt + \int_2^3 (t-2)dt$$

$$= \left[2t - \frac{t^2}{2}\right]_0^2 + \left[\frac{t^2}{2} - 2t\right]_2^3$$

$$= (4-2) + \left(\frac{9}{2} - 6\right) - (2-4)$$

$$2 + \frac{9}{2} - 6 + 2 = -2 + \frac{9}{2}$$

$$= \frac{9-4}{2} = \frac{5}{2}$$

4.) a.) Complete the blanks in the following statement of the Fundamental Theorem of Calculus.

Fundamental Theorem of Calculus:

Suppose f is continuous on $[a, b]$.
If $g(x) = \int_a^x f(t)dt$, then $g'(x) =$ $\underline{f(x)}$.
$\int_a^b f(x)dx =$ $\underline{F(b) - F(a)}$, where F is any antiderivative of f.

b.) Use the Fundamental Theorem of Calculus to evaluate the following.

i.) $\displaystyle \frac{d}{dy}\int_2^y \frac{\sin t}{t^2+3}dt$ $\qquad \dfrac{\sin y}{y^2+3}$

ii.) $\displaystyle \frac{d}{dx}\int_x^{x^4} \sqrt{t}dt$ $\qquad \sqrt{x^4}\cdot 4x^3 - \sqrt{x}$

$x^4 = u$

$$\int_x^{x^4}\sqrt{t}dt = \int_0^{x^4}\sqrt{t}dt - \int_0^x\sqrt{t}dt$$

$$\frac{d}{dx}\int_x^{x^4}\sqrt{t}dt = \sqrt{u}\cdot\frac{du}{dx} - \sqrt{x}$$

$$= x^2\cdot 4x^3 - \sqrt{x}$$

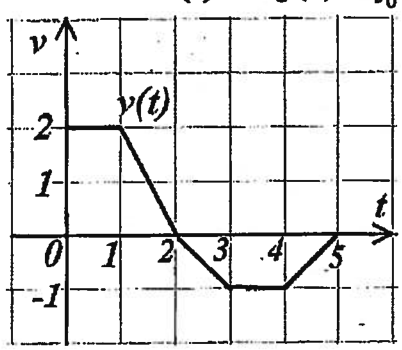
c.) Answer each of the following questions. No work or explanations are needed.

i.) If $f(t)$ is measured in dollars per year and t in years, what are the units of $\int_0^{10} f(t)\,dt$? dollars

ii.) True or False: All continuous functions have derivatives. false (~~log~~ $|x|$)

iii.) True or False: All continuous functions have antiderivatives. true $\int_0^x f(t)\,dt$

iv.) Below is the graph of a function $v(t)$. Let $g(x) = \int_0^x v(t)dt$.



Find each of the following:

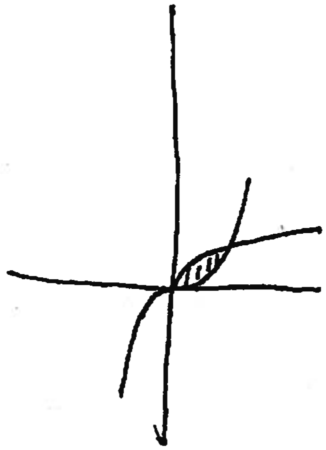
$g(0) =$ $\underline{0}$ $\quad g(2) =$ $\underline{3}$ $\quad g'(1) =$ $\underline{2}$ $\quad g'(4) =$ $\underline{-1}$

$2 + \frac{1}{2}\cdot 1\cdot 2$

$\underbrace{g'(1) = 2 \quad g'(4) = -1}_{\text{FTC I}}$

5.) Let S be the region bounded by $y = x^3$ and $y = \sqrt{x}$.

a.) Find the area of region S.



Intersect at

$$x^3 = \sqrt{x}$$

$$x = 0 \ \& \ x = 1$$

$$\int_0^1 (\sqrt{x} - x^3)\, dx$$

$$= \frac{2}{3} x^{\frac{3}{2}} - \frac{x^4}{4} \Big|_0^1$$

$$= \frac{2}{3} - \frac{1}{4}$$

Reality check: it is positive.

b.) i.) Find the volume obtained by revolving the region S about the x axis.

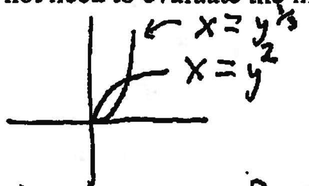
$$\int_0^1 \left[\pi\left(\sqrt{x}\right)^2 - \pi\left(x^3\right)^2\right]dx$$

$$= \int_0^1 \left[\pi x - \pi x^6\right]dx$$

$$= \left[\frac{\pi x^2}{2} - \frac{\pi x^7}{7}\right]_0^1$$

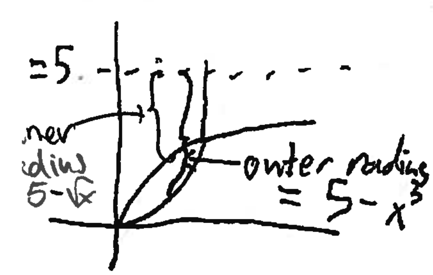
$$= \pi\left(\frac{1}{2} - \frac{1}{7}\right)$$

ii.) Set up an integral to find the volume obtained by revolving the region S about the y axis.
(You do not need to evaluate the integral.)



$$\int_0^1 \left[\pi\left(y^{\frac{1}{3}}\right)^2 - \pi\left(y^2\right)^2\right]dy$$

iii.) Set up an integral to find the volume obtained by revolving the region S about the line $y = 5$.
(You do not need to evaluate the integral.)



$$\cancel{\int_0^1 \pi\left(\sqrt{x}+5\right)^2 - \pi\left(x^3+5\right)^2 dx}$$

$$\int_0^1 \left[\pi\left(5-x^3\right)^2 - \pi\left(5-\sqrt{x}\right)^2\right]dx$$