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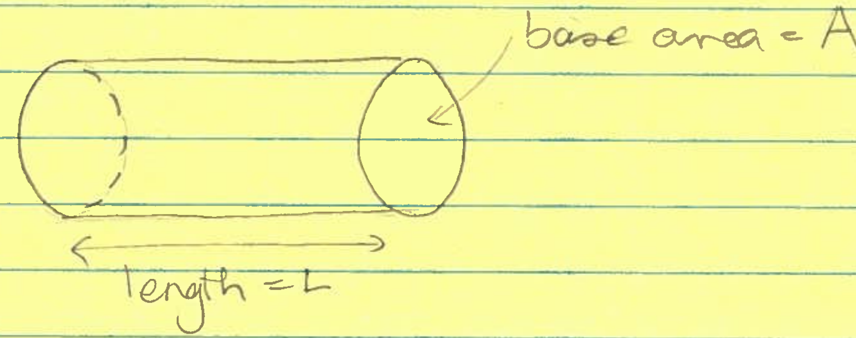
L8
6.2

Volumes §6.2

We can use calculus to compute the volumes of solids, much as we used it to compute areas of planar regions.

Suppose we wanted to compute the volume of a cylinder:



We know from high-school math that the volume is AL.

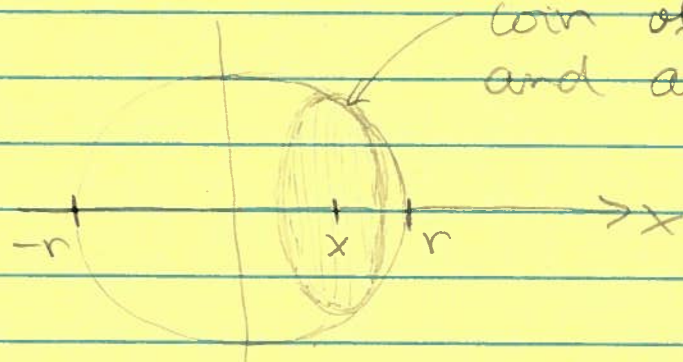
Now suppose the cylinder is actually a pile of N coins (glued together) each of volume v. Then another way to compute the volume would be Nv. This is the way you compute volumes in calculus!

But what if the "cross-sectional" area changes as a function of x? That is, what if the coins are not all the same size? This is where integration comes in:

EX Show that the volume of a sphere is $\frac{4}{3}\pi r^3$ using calculus.

Sol^{n}

coin of thickness dx and area $A(x)$.

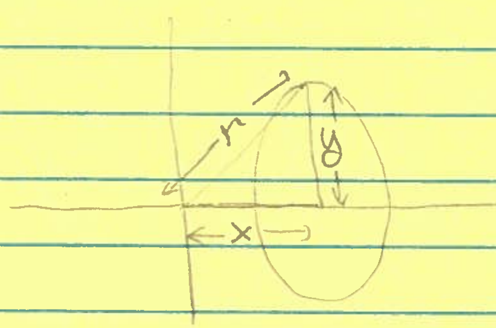


volume of sphere = sum of volumes of coins

$$= \int_{-r}^{r} A(x)\,dx$$

$$= 2\int_{0}^{r} A(x)\,dx. \qquad \text{(symmetry)}$$

But what is $A(x)$? Radius of coin is y where:



$$\Rightarrow r^2 = x^2 + y^2$$

$$\Rightarrow A(x) = \pi y^2 = \pi(r^2 - x^2)$$

Thus volume of sphere is:

$$2\int_0^r \pi(r^2-x^2)\,dx$$

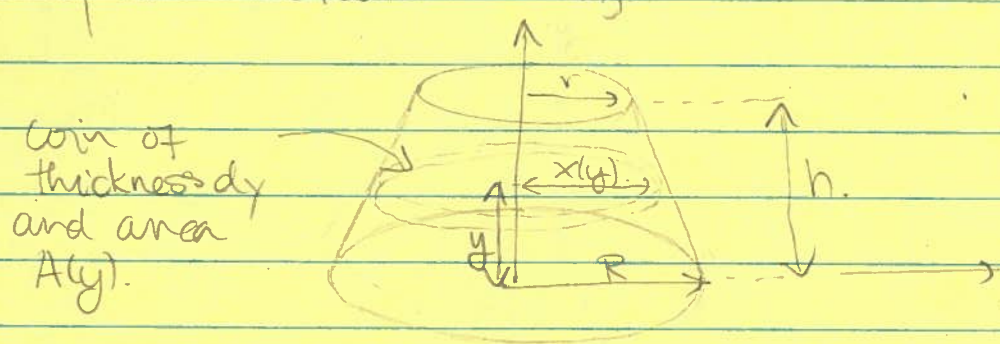
$$= 2\pi\cdot\left[r^2x - \frac{x^3}{3}\right]_0^r$$

$$= 2\pi\left(r^3 - \frac{r^3}{3}\right)$$

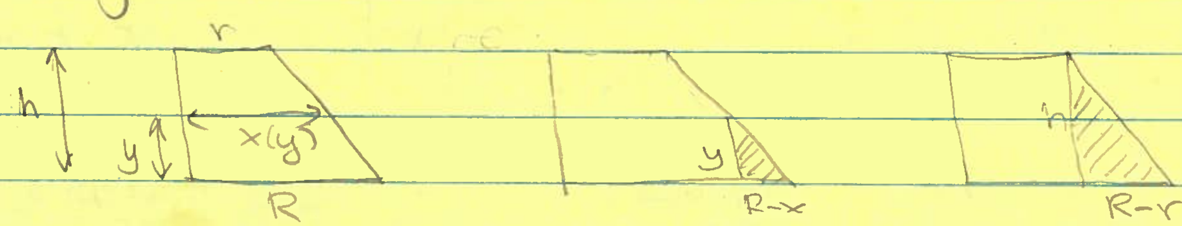
$$= 2\pi\,\frac{2}{3}r^3 = \frac{4}{3}\pi r^3$$

Other "coin" problems:

Ex Compute volume of:



Sol^n The trick here is to find similar triangles in the cross section



similarity of shaded triangles $\Rightarrow$

$$\frac{y}{R-x} = \frac{h}{R-r}$$

$$\Rightarrow \quad h(R-x) = y(R-r)$$

$$\Rightarrow \quad hR - y(R-r) = hx.$$

$$\Rightarrow \quad x = R - \frac{R-r}{h}y.$$

Thus area of coin at height y is:

$$A(y) = \pi x^2$$
$$= \pi\left(R - \frac{R-r}{h}y\right)^2$$

and volume of frustrum is:

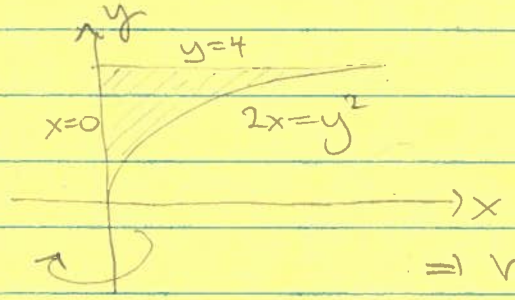
$$\int_0^h dy\, A(y) = \int_0^h dy \cdot \pi\left(R - \frac{R-r}{h}y\right)^2$$

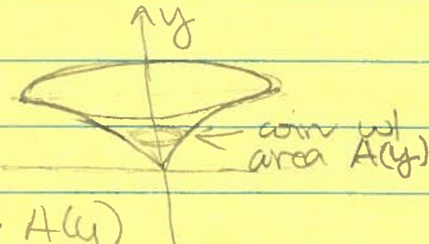
<u>EX</u> Find the volume of the solid obtained by rotating the region bounded by the curve's

$$2x = y^2, \quad x = 0, \quad y = 4$$

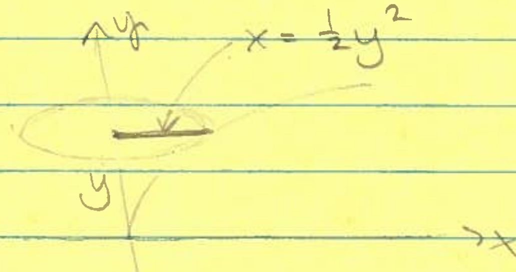
about the y-axis.

Sol^n





$\Rightarrow$ volume = $\int_0^4 dy\, A(y)$



Area of coin:

$$A(y) = \pi x^2$$
$$= \pi\, y^4/4$$

$$\Rightarrow \text{volume} = \int_0^4 dy\, \frac{\pi}{4} y^4 = \frac{\pi}{4} \frac{y^5}{5} \Big]_0^4 = \frac{\pi}{5} 4^4$$

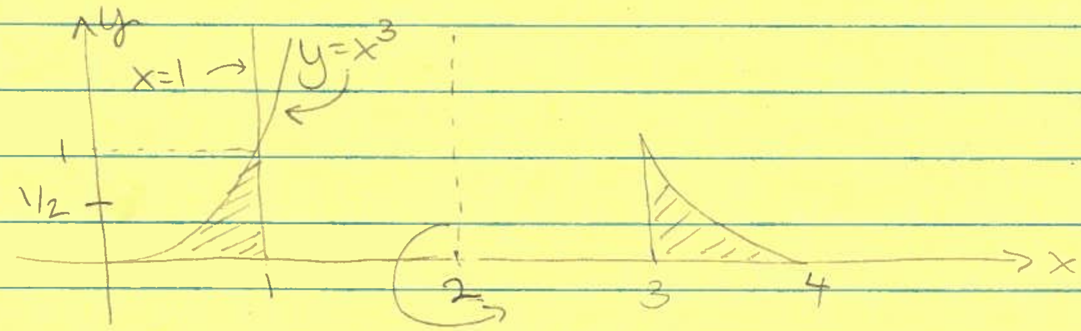
WASHER PROBLEM:

EX Find volume of solid obtained by rotating the region bounded by

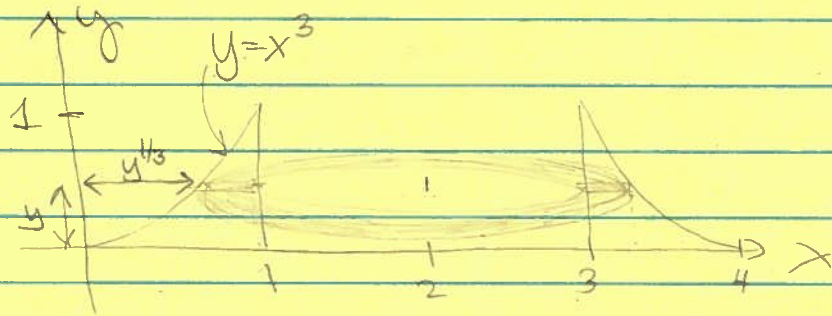
$$y = x^3, \quad y = 0, \quad x = 1$$

about $x = 2$.

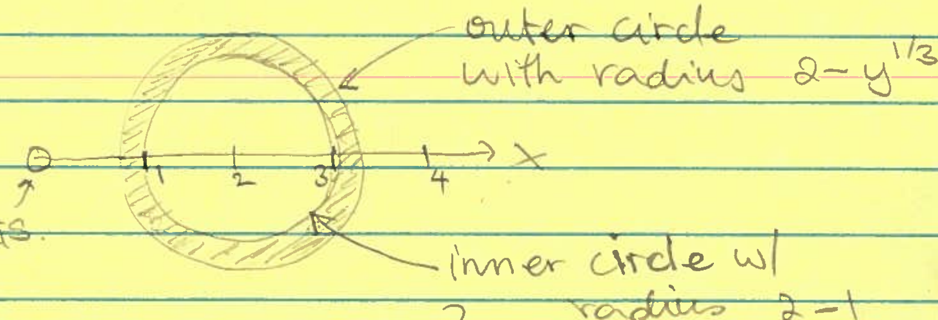
Sol^n



If we slice through the solid using the plane $y = 1/2$, say, then the cross-section would look like a washer:



Looking down on the washer from above, we would see:



What is its area $A(y)$?

$$A(y) = (\text{area of outer circle}) - (\text{area of inner circle})$$

$$= \pi(2-y^{1/3})^2 - \pi(2-1)^2$$

Thus:

$$\text{volume of solid} = \int_0^1 A(y)\,dy$$

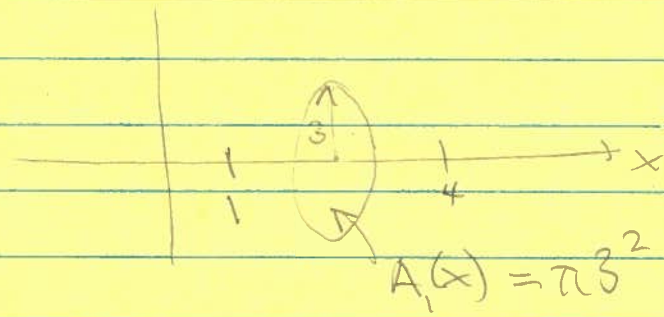
$$= \int_0^1 \pi\left[(2-y^{1/3})^2 - 1\right] dy$$

<u>EX</u> Describe the solid whose volume is

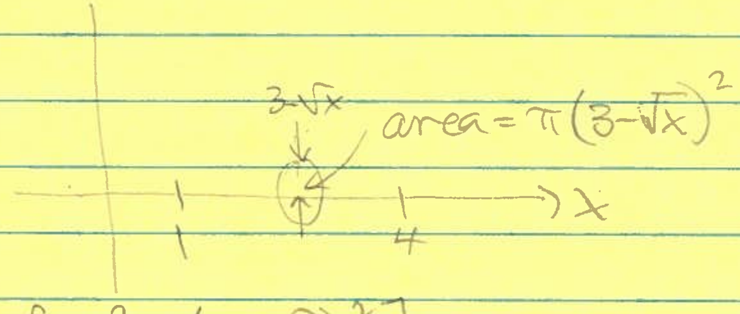
$$\pi \int_1^4 [3^2 - (3-\sqrt{x})^2]\, dx$$

<u>Sol^n</u> The trick is to notice the π and the square and remember that coin areas are of the form πr^2.

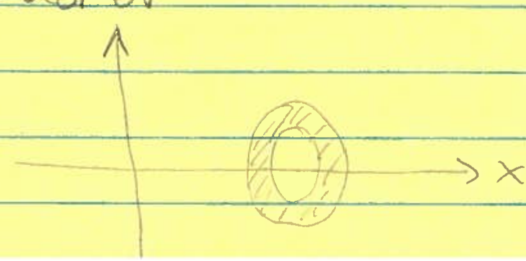
Imagine a coin for each x with radius 3:



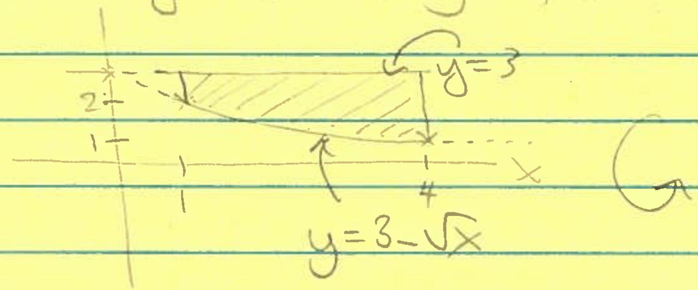
and another coin for the same x with radius $3-\sqrt{x}$:



Then $\pi[3^2 - (3-\sqrt{x})^2]$ represents area of the washer:



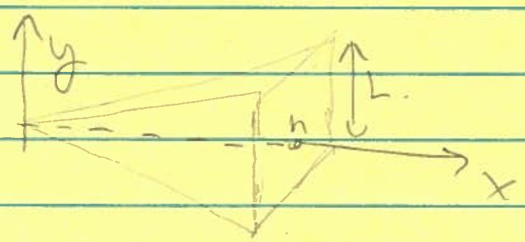
and the washer is part of the solid obtained by rotating following region about x-axis:

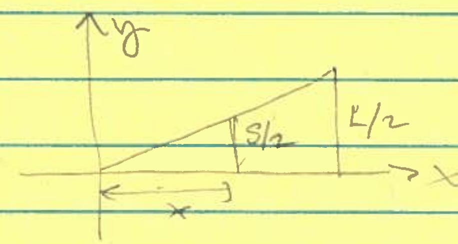


We can also do solids that are not solids of revolution:

Ex Find the volume of a pyramid whose base is square w/ side L and whose height is h.

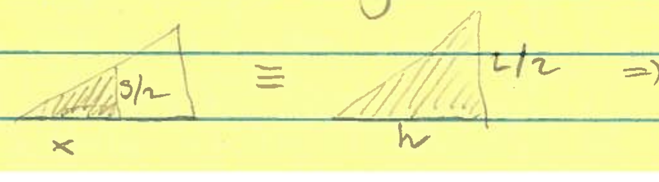
Sol^n





cross section of pyramid is square w/ area s^2; we would like this as a function of x.

Similar triangles



$$\frac{s/2}{x} = \frac{L/2}{h}.$$

$$\Rightarrow \quad s = \frac{Lx}{h} \qquad \Rightarrow \quad A(x) = s^2 = \frac{L^2 x^2}{h^2}$$

$$\Rightarrow \text{volume} = \int_0^h dx\, A(x)$$

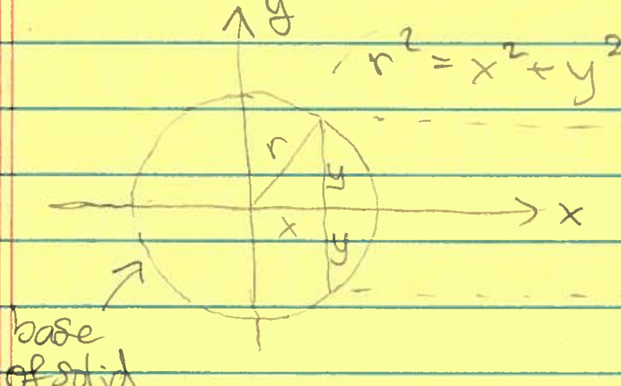
$$= \int_0^h dx\, \frac{L^2 x^2}{h^2}$$

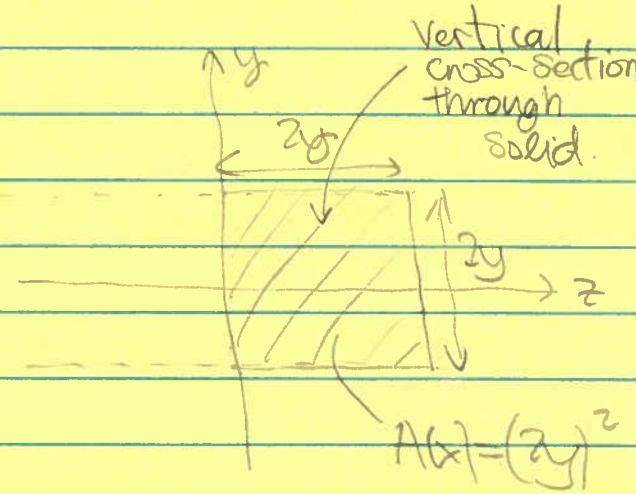
$$= \frac{L^2}{h^2} \frac{x^3}{3} \Big]_0^h$$

$$= \frac{L^2}{h^2} \frac{h^3}{3} = \frac{1}{3} L^2 h$$

Ex Base of solid is circle with radius r. Parallel cross-sections $\perp$ to base are squares. Compute volume of solid.

Sol^n:





$$\text{volume} = \int_{-r}^{r} dx\, A(x)$$

$$= 2\int_0^r dx\, 4(r^2 - x^2)$$

$$= 8\left[r^2 x - \frac{x^3}{3}\right]_0^r$$

$$= 8\left(r^3 - \frac{r^3}{3}\right)$$

$$= \frac{16}{3} r^3$$