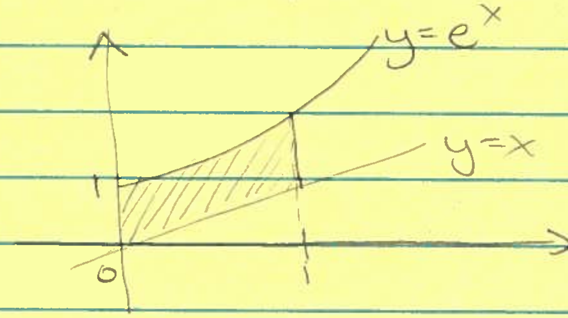
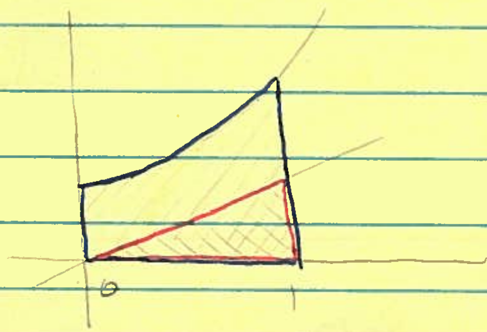
L7
§6.1, 6.2

Areas between curves (6.1)

Ex Find the area of the following region:



Sol^n The trick is to view the area as the difference of two other areas:



Requested area

$$= \text{area}(\;) - \text{area}(\;)$$

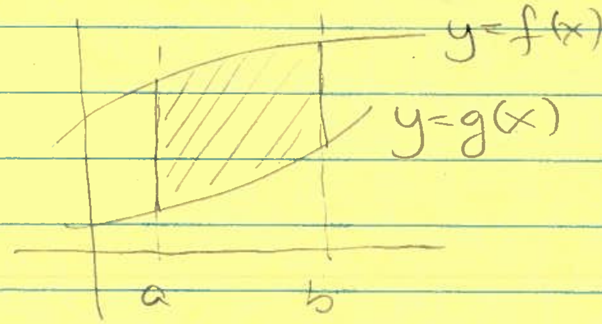
$$= \int_0^1 e^x\,dx - \int_0^1 x\,dx$$

$$= \int_0^1 (e^x - x)\,dx \qquad (1)$$

$$= \left[e^x - \frac{x^2}{2}\right]_0^1 = \left(e - \tfrac{1}{2}\right) - \left(e^0 - 0\right)$$

$$= e - 3/2$$

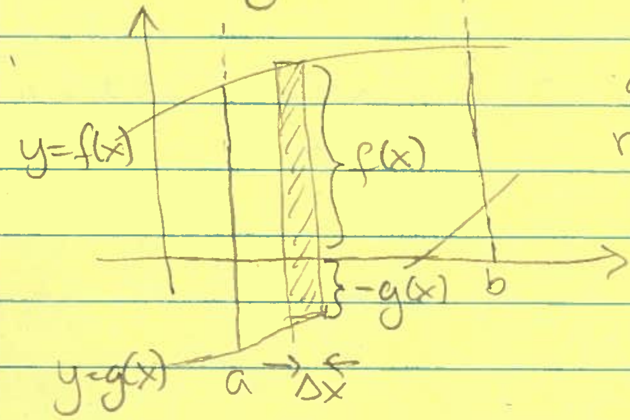
This example illustrates an important point:



$$\text{area} = \int_a^b [f(x) - g(x)]\,dx \quad (*)$$

[cf. (1) on p1.]

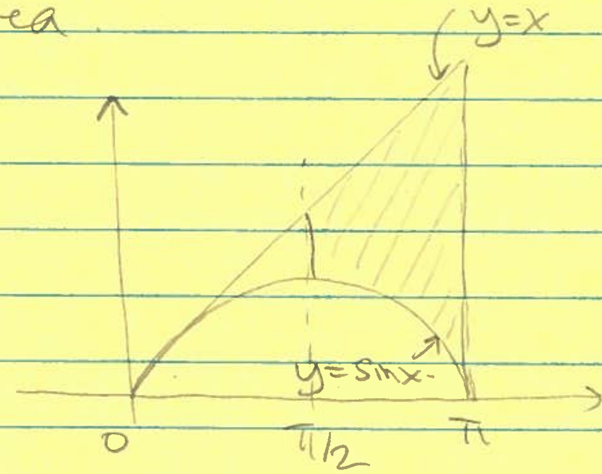
This holds even if $g(x) < 0$ [provided $f(x) \geq g(x)$]:



area of Riemann rectangle $= [f(x) - g(x)]\Delta x$

EX Sketch the region enclosed by $y = \sin x$, $y = x$, $x = \pi/2$, and $x = \pi$. Find its area.

Sol^n



$y = \sin x \approx x - \frac{x^3}{3!} + \cdots$

$\Rightarrow$ $y = \sin x$ looks like $y = x$ when $x \ll 1$.

$$\text{Area} = \int_{\pi/2}^{\pi} (x - \sin x)\,dx = \left[\frac{x^2}{2} + \cos x\right]_{\pi/2}^{\pi}$$

$$= \left[\frac{\pi^2}{2} + \cos\pi\right] - \left[\frac{(\pi/2)^2}{2} + \cos\left(\frac{\pi}{2}\right)\right]$$

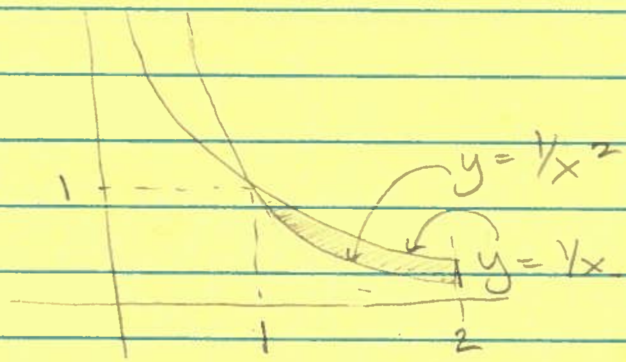
$$= \frac{\pi^2}{2} - 1 - \frac{\pi^2}{8}$$

$$= \frac{3\pi^2}{8} - 1.$$

Ex Compute area of region defined by

$$y = 1/x,\ y = 1/x^2,\ x = 1,\ x = 2$$

Sol^n



$x > 1 \Rightarrow x^2 > x$

$\Rightarrow \frac{1}{x^2} < \frac{1}{x}$

$$\text{area} = \int_1^2 \left(\frac{1}{x} - \frac{1}{x^2}\right) dx$$

$$= \left[\ln x - \frac{x^{-1}}{-1}\right]_1^2 = \left[\ln x + \frac{1}{x}\right]_1^2$$

$$= \left[\ln 2 + \frac{1}{2}\right] - \left[\cancel{\ln 1} + \frac{1}{1}\right]$$

$$= \ln 2 - \frac{1}{2}$$

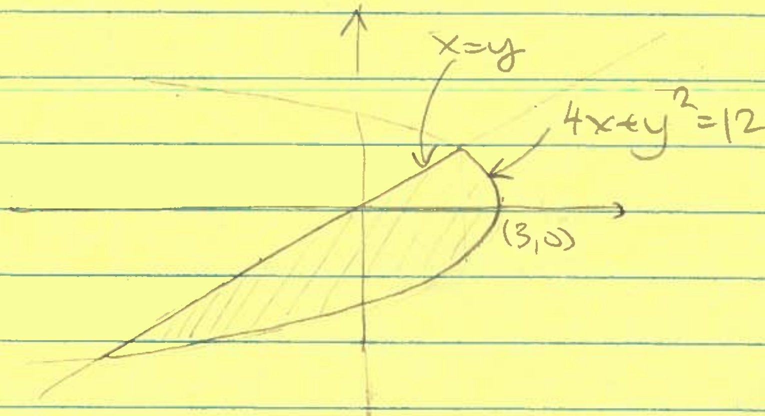
EX Compute area of region defined by:

$$4x + y^2 = 12, \quad x = y.$$

Sol^n $4x + y^2 = 12$

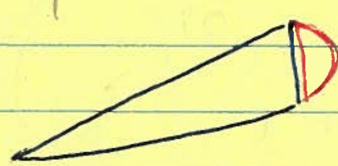
$\Rightarrow 4x = 12 - y^2$

$\Rightarrow x = 3 - y^2/4$



Note: top boundary consists of two curves.

$\Rightarrow$ To compute the area, we must divide the region in two in such a way that top and bottom boundaries consist of single curves...

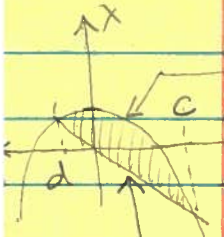


Then area of entire region is:

area() + area(D)

There is a quicker way to solve this problem!

The trick is to view the curves not as functions of x, but as functions of y. When we do this, the top



boundary is a single curve ($x = f(y) = 3 - y^2/4$) and the bottom boundary is another ($x = g(y) = y$). Thus, according to (*) on p 2,

$$\text{area} = \int_c^d [f(y) - g(y)]\,dy$$

$$= \int_c^d \left[\left(3 - \frac{y^2}{4}\right) - y\right] dy$$

To find c and d we must find the y-coordinates of the points of intersection of the curves, ie. solve $f(y) = g(y)$:

$$3 - y^2/4 = y$$

$$\Rightarrow 12 - y^2 = 4y$$

$$\Rightarrow y^2 + 4y - 12 = 0$$

$$\Rightarrow (y+6)(y-2) = 0$$

$$\Rightarrow c = -6, \quad d = 2$$

Thus:

$$\text{area} = \int_{-6}^{2} \left[[3 - y^2/4] - y \right] dy$$

$$= \int_{-6}^{2} (3 - y^2/4 - y)\, dy$$

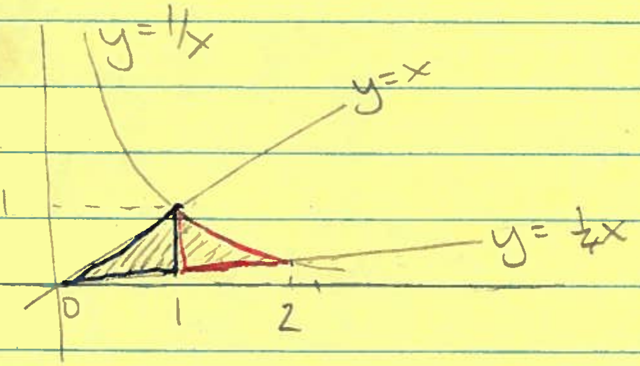
$$= \left[3y - \frac{y^3}{12} - \frac{y^2}{2} \right]_{-6}^{2}$$

$$= \left[6 - \frac{2}{3} - 2 \right] - \left[\cancel{-18} + \cancel{18} - 18 \right]$$

$$= 3\tfrac{1}{3} + 18 = \frac{10}{3} + \frac{54}{3} = \frac{64}{3}$$

EX Compute area between $y = 1/x$, $y = x$, $y = \frac{1}{4}x$. Suppose $x > 0$.

Sol^{n}



$y = 1/x$ and $y = \frac{1}{4}x$ intersect when

$$\frac{1}{x} = \frac{1}{4}x$$

$$\Rightarrow 4 = x^2$$

$$\Rightarrow x = 2$$

area = area() + area()

$$= \int_0^1 \left(x - \tfrac{1}{4}x\right) dx + \int_1^2 \left(\tfrac{1}{x} - \tfrac{1}{4}x\right) dx$$

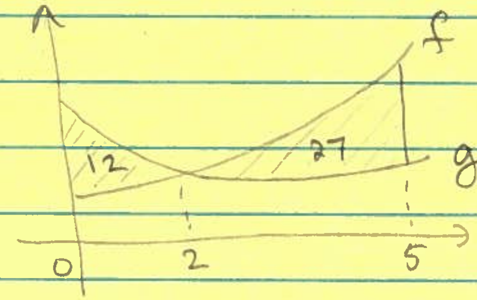
$$= \int_0^1 \tfrac{3}{4} x\, dx + \quad - \text{''} -$$

$$= \tfrac{3}{4} \frac{x^2}{2} \Big]_0^1 + \left[\ln x - \tfrac{1}{4}\frac{x^2}{2} \right]_1^2$$

$$= \cancel{\tfrac{3}{8}} + \left[\ln 2 - \cancel{\tfrac{1}{2}} \right] - \left[\cancel{\ln 1} - \cancel{\tfrac{1}{8}} \right]$$

$$= \ln 2$$

EX Consider:



Total area $= 12 + 27 = 39$ but:

$$\int_0^5 [f(x) - g(x)]\,dx$$

$$= \int_0^2 [f(x) - g(x)]\,dx + \int_2^5 [f(x) - g(x)]\,dx$$

$$= -\int_0^2 \underbrace{[g(x) - f(x)]}_{g > f}\,dx + \int_2^5 \underbrace{[f(x) - g(x)]}_{f > g}\,dx$$

$$= -12 + 27$$

$$= 15.$$