2B

L6 §5.5

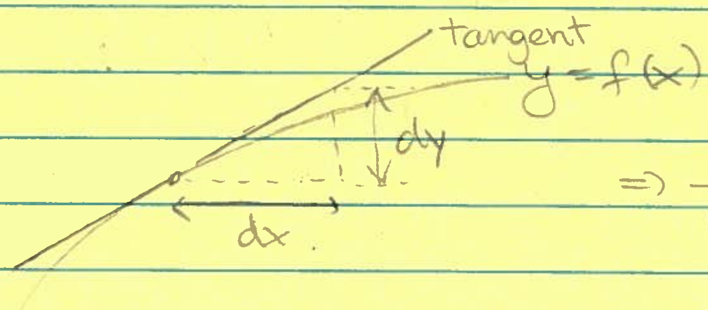
Substitution Rule

Recall from §3.10:

DIFFERENTIAL

$$\frac{dy}{dx} = f'(x) \qquad (*)$$

dy and dx are "differentials" and may be treated as numbers.
This leads to the following geometric interpretation:



$\Rightarrow \frac{dy}{dx}$ = slope of tangent.

Now multiply (*) across by dx to get:

$$dy = f'(x)\,dx$$

Aside { The utility of dy is that it is a good approx for $\Delta y = f(x+dx) - f(x)$ when dx is small. See picture above.

We will use the differential to help us compute integrals...

Consider

$$I = \int 2x\sqrt{1+x^2}\,dx$$

Make the substitution

$$u = 1+x^2 = f(x)$$

$$\Rightarrow \quad du = f'(x)\,dx$$

$$= 2x \cdot dx$$ ... THESE FACTORS OCCUR IN INTEGRAND!

Thus:

$$I = \int \sqrt{1+x^2} \cdot 2x\,dx$$

$$= \int u^{1/2}\,du$$ ... SIMPLER INTEGRAL!

$$= \frac{u^{3/2}}{3/2} + c$$

$$= \frac{2}{3}(1+x^2)^{3/2} + c.$$

This is the substition technique (rule). We can check its validity:

$$\frac{d}{dx}\left[\frac{2}{3}(1+x^2)^{3/2} + c\right] \underset{\text{chain rule}}{=} \frac{2}{3}\,\frac{3}{2}(1+x^2)^{1/2} \cdot 2x$$

$$= 2x\sqrt{1+x^2}$$

which is the integrand of I. This is

as it should be, since

$$\int f(x)\,dx = F(x)$$

where $F' = f$ (cf. 25).

Ex Evaluate:

$$\int x^3 \underbrace{\sqrt{x^2+1}}_{u}\,dx$$

Sol^n

$$du = 2x\,dx \quad , \quad x^2 = u-1$$

$$\int x^3\sqrt{x^2+1}\,dx = \int x^2\sqrt{x^2+1}\;x\,dx$$

$$= \int (u-1)\sqrt{u}\;\tfrac{1}{2}du$$

$$= \frac{1}{2}\int\left(u^{3/2} - u^{1/2}\right)du$$

$$= \frac{1}{2}\left(\frac{u^{5/2}}{5/2} - \frac{u^{3/2}}{3/2}\right) + c$$

$$= \frac{1}{5}(x^2+1)^{5/2} - \frac{1}{3}(x^2+1)^{3/2} + c$$

Alternative Sol^n

$$u = \sqrt{x^2+1} \Rightarrow u^2 = x^2+1.$$

$$\Rightarrow 2u\,du = 2x\,dx$$

and $x^2 = u^2 - 1$

$$\int x^3\sqrt{y^2+1}\,dx = \int x^2\sqrt{x^2+1}\;x\,dx$$

$$= \int (u^2-1)\cdot u \cdot u\,du$$

$$= \int (u^4 - u^2)\,du$$

$$= \frac{u^5}{5} - \frac{u^3}{3} + C = \frac{(x^2+1)^{5/2}}{5} - \frac{(x^2+1)^{3/2}}{3} + C.$$

as before.

EX $\displaystyle\int \frac{x\,dx}{1+x^4}$

$$= \int \frac{\frac{1}{2}du}{1+u^2} \qquad \begin{aligned} u &= x^2 \\ du &= 2x\,dx \end{aligned}$$

$$= \frac{1}{2}\arctan u + C = \frac{1}{2}\arctan(x^2) + C.$$

ASIDE: $\displaystyle\frac{d}{du}\arctan u = \frac{1}{1+u^2}$

To see this write:

$$\arctan u = \theta$$

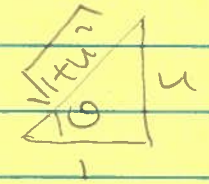
$$\Rightarrow \tan\theta = u$$

$$\Rightarrow \frac{d}{du}\tan\theta = \frac{d}{du}\cdot u$$

$$\Rightarrow \underbrace{\frac{1}{\cos^2\theta}}_{\text{cf p. 225}} \cdot \frac{d\theta}{du} = 1.$$

$$\Rightarrow \frac{d\theta}{du} = \cos^2\theta.$$

But $\tan\theta = u \Rightarrow$



$$\Rightarrow \cos^2\theta = \frac{1}{1+u^2}$$

Thus: $\dfrac{d\theta}{du} = \dfrac{1}{1+u^2}$, i.e. $\dfrac{d}{du}\arctan u = \dfrac{1}{1+u^2}$ $\boxed{\times}$

<u>EX</u> $\displaystyle\int \frac{\cos(\ln t)}{t}\,dt$ $\qquad u = \ln t$

$du = \frac{1}{t}\,dt$.

$$= \int \cos(\ln t)\,\frac{dt}{t}$$

$$= \int \cos u \cdot du = \sin u + C = \sin(\ln t) + C.$$

We may also use the substitution rule for definite integrals. It is often preferable in those cases to "substitute" the limits of integration, along with the variable of integration:

<u>EX</u> $\displaystyle\int_1^2 x\sqrt{x-1}\,dx$ $\qquad u = x-1$

$u+1 = x$

$$= \int_1^2 \sqrt{x-1}\cdot x\cdot dx \qquad du = dx$$

$$= \int_0^1 \sqrt{u}\cdot(u+1)\cdot du \qquad \begin{array}{l} x=1 \Rightarrow u=0 \\ x=2 \Rightarrow u=1 \end{array}$$

$$= \int_0^1 (u^{3/2} + u^{1/2})\,du$$

$$= \left[\frac{u^{5/2}}{5/2} + \frac{u^{3/2}}{3/2}\right]_0^1 = \frac{2}{5} + \frac{2}{3} = \frac{6+10}{15} = \frac{16}{15}.$$

<u>EX</u> $\displaystyle\int_1^2 \frac{e^{1/x}}{x^2}\,dx$ $\qquad u = \frac{1}{x}$; $du = -\frac{1}{x^2}dx$

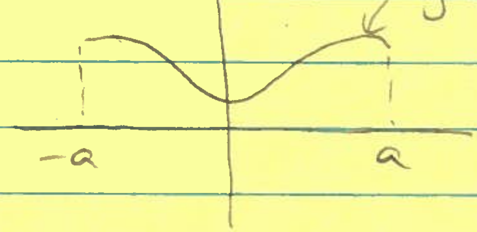
$x = 1 \Rightarrow u = 1$
$x = 2 \Rightarrow u = 1/2$

$$= -\int_1^{1/2} e^u\,du$$

$$= \int_{1/2}^1 e^u\,du = e^u\Big|_{1/2}^1 = e^1 - e^{1/2} = e - \sqrt{e}$$

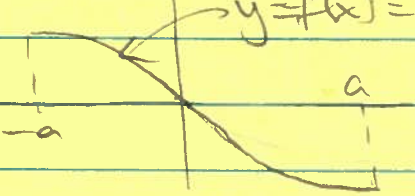
We may use the substitution rule to show that

$y = f(x) =$ "even"



$$\int_{-a}^{a} f(x)\,dx = 2\int_0^a f(x)\,dx$$

$y = f(x) =$ "odd"!



$$\int_{-a}^{a} f(x)\,dx = 0$$

Always check to see if your integrand is "even" or "odd"!

One last example to keep you on your toes!

Ex Compute $\int_0^3 x f(x^2)\,dx$

if $\int_0^9 f(x)\,dx = 4$

Sol^n $u = x^2 \Rightarrow du = 2x\,dx$

$\Downarrow$

$x = 0 \Rightarrow u = 0$
$x = 3 \Rightarrow u = 9$

$$\int_0^3 x f(x^2)\,dx$$
$$= \int_0^3 f(x^2)\cdot x\,dx$$
$$= \int_0^9 f(u)\cdot \tfrac{1}{2}\,du$$
$$= \frac{1}{2}\int_0^9 f(u)\,du$$
$$= \frac{1}{2}\int_0^9 f(x)\,dx$$

x, u are "dummy" variables.

$$= \frac{1}{2}(4)$$
$$= 2.$$