2B

L5
§5.4

Indefinite Integrals and the Net Change Theorem

Recall: FTC2 says:

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b \quad (= F(b) - F(a)) \quad (*)$$

where $F(x)$ is an antiderivative of f, ie $F'(x) = f(x)$.

The anti-derivative is usually called the "indefinite integral" and written:

$$\int f(x)\,dx := F(x)$$

The reason for this nomenclature is that $(*)$ can now be written

$$\int_a^b f(x)\,dx = \int f(x)\,dx\Big|_a^b .$$

EX. $\int \left(\frac{1+r}{r}\right)^2 dr = \int \frac{1+2r+r^2}{r^2}\,dr = \int \left(r^{-2} + \frac{2}{r} + 1\right) dr$

$= -r^{-1} + 2\ln r + r + C$

EX: $I = \int (2 + \tan^2\theta)\,d\theta = \int [1 + (1 + \tan^2\theta)]\,d\theta$

But:

$$1+\tan^2\theta = 1+\frac{\sin^2\theta}{\cos^2\theta} = \frac{\overbrace{\cos^2\theta+\sin^2\theta}^{1}}{\cos^2\theta}$$

$$= \frac{1}{\cos^2\theta}$$

Thus

$$I = \int \left(1+\frac{1}{\cos^2\theta}\right) d\theta$$

Now:

$$\int \frac{1}{\cos^2\theta}\, d\theta = \tan\theta + C$$

ASIDE: To see this, differentiate RHS:

$$\frac{d}{d\theta}(\tan\theta + C) = \frac{d}{d\theta}(\tan\theta) = \frac{d}{d\theta}\left(\frac{\sin\theta}{\cos\theta}\right)$$

$$\underset{\text{product rule}}{=} \frac{d}{d\theta}(\sin\theta)\frac{1}{\cos\theta} + \sin\theta\,\frac{d}{d\theta}\left(\frac{1}{\cos\theta}\right)$$

$$= \cos\theta\,\frac{1}{\cos\theta} + \sin\theta\left[\underbrace{-\frac{1}{\cos^2\theta}\cdot(-\sin\theta)}_{\text{chain rule}}\right]$$

$$= 1+\frac{\sin^2\theta}{\cos^2\theta} = \frac{1}{\cos^2\theta}$$

Thus

$$I = \theta + \tan\theta + C.$$

<u>EX</u> $$\int_0^1 (x^{10} + 10^x)\,dx = \left[\frac{x^{11}}{11} + \frac{10^x}{\ln 10}\right]_0^1$$

$$= \left[\frac{1^{11}}{11} + \frac{10^1}{\ln 10}\right] - \left[\frac{\cancel{0^{11}}}{\cancel{11}} + \frac{10^0}{\ln 10}\right]$$

$$= \frac{1}{11} + \frac{10}{\ln 10} - \frac{1}{\ln 10} = \frac{1}{11} + \frac{9}{\ln 10}.$$

<u>EX</u> Compute:

$$I = \int_0^{\sqrt{3}/2} \frac{dy}{\sqrt{1-y^2}}.$$

It turns out that:

$$\int \frac{dy}{\sqrt{1-y^2}} = \arcsin(y) + C.$$

ASIDE: To see this, compute

$$\frac{d}{dy} \arcsin(y)$$

by first $\underbrace{\text{defining}}$

$$\theta = \arcsin(y)$$

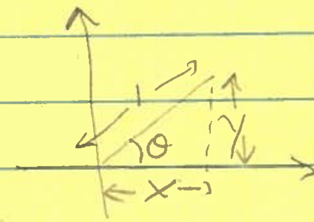
i.e. $\sin\theta = y$.

and then applying d/dy

$$\underbrace{\cos\theta \cdot \frac{d\theta}{dy}}_{\text{chain rule}} = 1 \Rightarrow \frac{d\theta}{dy} = \frac{1}{\cos\theta} \quad (1)$$

But: $\sin\theta = y$ $\Longrightarrow$

$\Longleftarrow$



$\cos\theta = x$

$= \sqrt{1-y^2}$ (by Pythagoras)

Thus: (1) $\Rightarrow$

$$\frac{d}{dy}\arcsin(y) = \frac{1}{\sqrt{1-y^2}}.$$

$$\Rightarrow \quad \int \frac{dy}{\sqrt{1-y^2}} = \arcsin(y) + C.$$

Finally:

$$I = \Big[\arcsin(y)\Big]_0^{\sqrt{3}/2}$$

(the "C" needn't be included because it drops out)

$$= \arcsin(\sqrt{3}/2) - \arcsin(0)$$

$$= \frac{\pi}{3} - 0$$

because $\sin(\pi/3) = \sqrt{3}/2$, $\sin(0) = 0$.

Recall FTC2 :

$$\int_a^b F'(x)\,dx = F(b) - F(a)$$

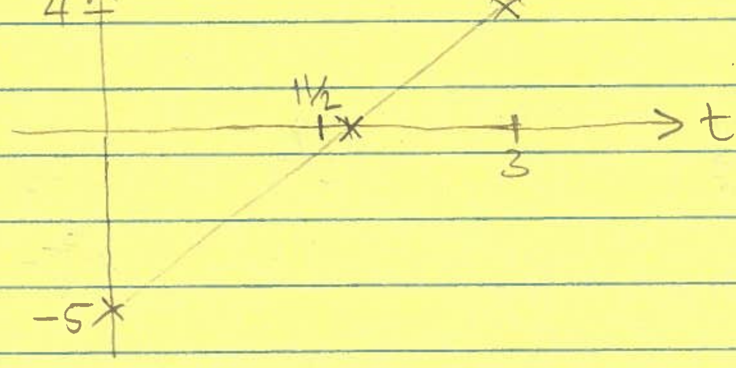
FTC2 is sometimes called the "net-change theorem" because $F'(x)$ is the rate @ which $y = F(x)$ changes w.r.t. x, while $F(b) - F(a)$ is the net change in F between $x = a$ and $x = b$.

Q. A particle moves along a line with velocity

$$v(t) = 3t - 5 \qquad 0 < t \leq 3$$

Find the displacement of the object.

A. In this example v takes on +ve and -ve values:



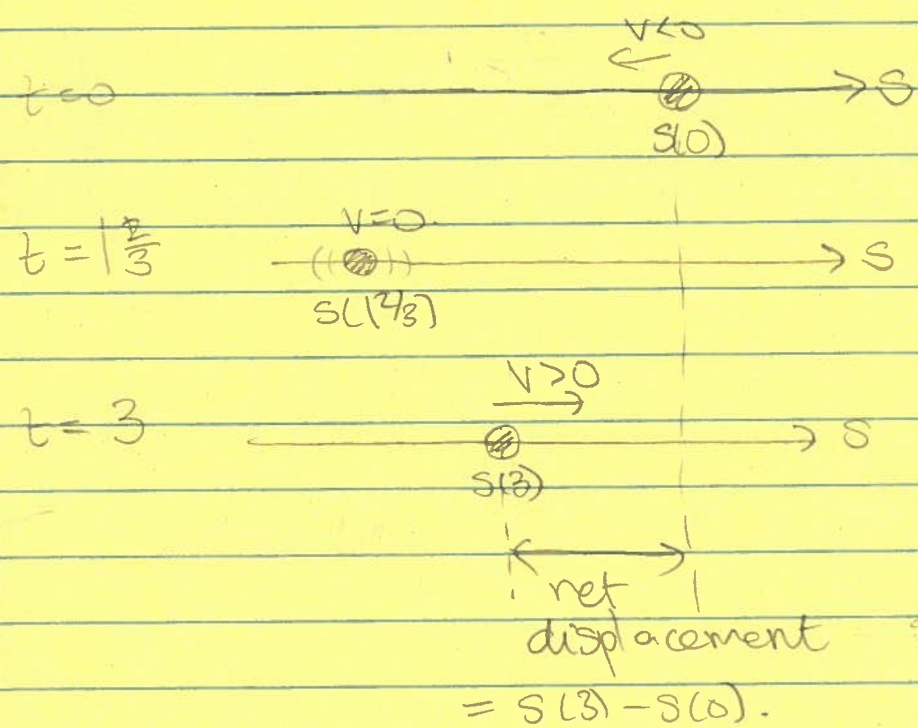
When $v<0$ the particle moves to the left:

$$\xleftarrow{v<0}\bullet \qquad 0 \leq t \leq 1\tfrac{2}{3}$$

When $v>0$, it moves right-ward:

$$\bullet\xrightarrow{v>0} \qquad 1\tfrac{2}{3} \leq t \leq 3.$$

Let's track the position, $s(t)$, of the particle @ various times, t:



$= s(3) - s(0).$

FTC 2 $\Rightarrow$

$$s(3) - s(0) = \int_0^3 s'(t)\,dt$$

But:

$$s'(t) = v(t)$$
$$= 3t-5$$

Thus

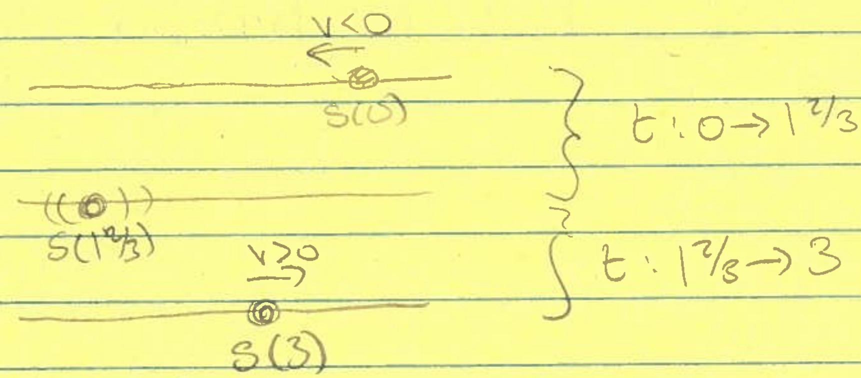
$$s(3)-s(0) = \int_0^3 (3t-5)\,dt$$

$$= \left[3\frac{t^2}{2} - 5t\right]_0^3$$

$$= \frac{27}{2} - 15 = \frac{27-30}{2} = -\frac{3}{2}.$$

Q. Find the distance travelled.

A. Recall:



In the first time interval, the particle moves in the -ve direction, ie. $v<0$. Thus the distance travelled during this time is:

$$s(0) - s(1\tfrac{2}{3}) = -\left[s(1\tfrac{2}{3}) - s(0)\right]$$

$$\overset{FTC2}{=} -\int_0^{1\frac{2}{3}} v(t)\,dt$$

$$= \int_0^{1\frac{2}{3}} (5-3t)\,dt$$

$$= \left[5t - 3\frac{t^2}{2}\right]_0^{1\frac{2}{3}}$$

$$= 5\frac{5}{3} - \frac{3}{2}\left(\frac{5}{3}\right)^2$$

In the second time interval ($1\frac{2}{3} \to 3$), the particle moves in the +ve direction ie. $v>0$. Thus distance travelled during this time interval is:

$$s(3) - s(1\tfrac{2}{3}) \overset{FTC2}{=} \int_{1\frac{2}{3}}^{3} v(t)\,dt$$

$$= \int_{1\frac{2}{3}}^{3} (3t-5)\,dt$$

$$= \left[3\frac{t^2}{2} - 5t\right]_{1\frac{2}{3}}^{3}$$

= you can work it out!