2B

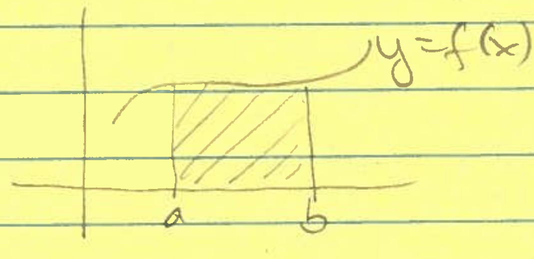
L4
§5.3

Fundamental Theorem of Calculus.

In L3, we encountered the definite integral,

$$\int_a^b f(x)dx,$$

and interpreted it as the area under the graph of $f(x)$ from $x=a$ to $x=b$:

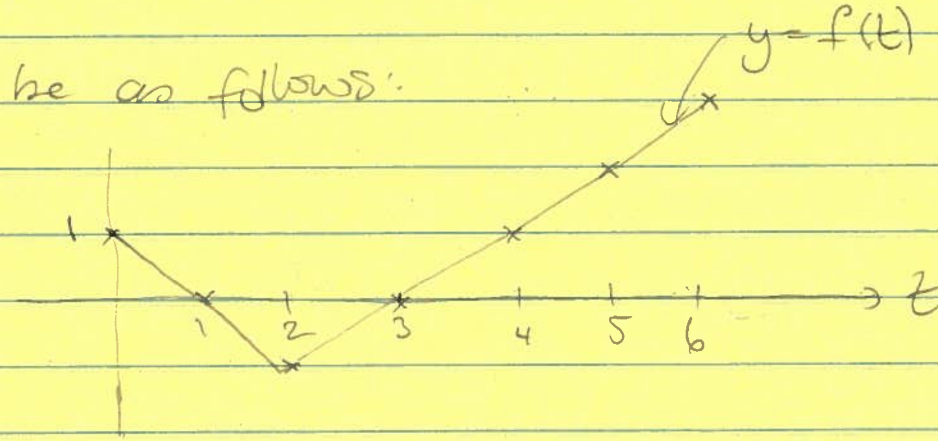


Indeed this is how we computed the definite integral. Today, we introduce the most important theorem in calculus, the Fundamental Theorem of Calculus (FTC), and use it to develop a much shorter method for computing the definite integral.

We begin by noting that the definite integral, which up to now is just a number, can be converted into a function as follows:

$$g(x) := \int_a^x f(t)dt.$$

Ex Let f be as follows:

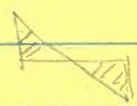


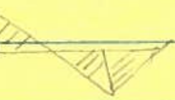
Compute $g(x)$ for $x = 0, 1, 2, \ldots, 6$

Sol^n

$g(0) = 0$

$g(1) =$  $= \frac{1}{2}$

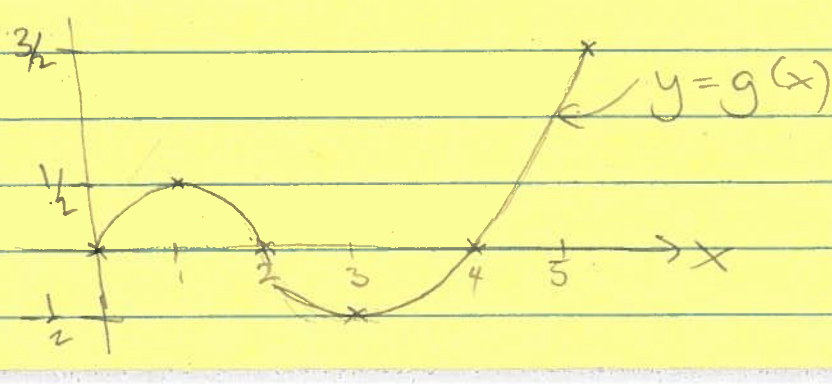
$g(2) =$  $= \frac{1}{2} - \frac{1}{2} = 0.$

$g(3) =$  $= \frac{1}{2} - 1 = -\frac{1}{2}$

$g(4) =$  $= \frac{1}{2} - 1 + \frac{1}{2} = 0.$

$g(5) =$  $=$ area() $= 1 + \frac{1}{2} = 3/2$

$g(6) =$  $=$ area() $= 2 + \frac{1}{2}(4) = 4$



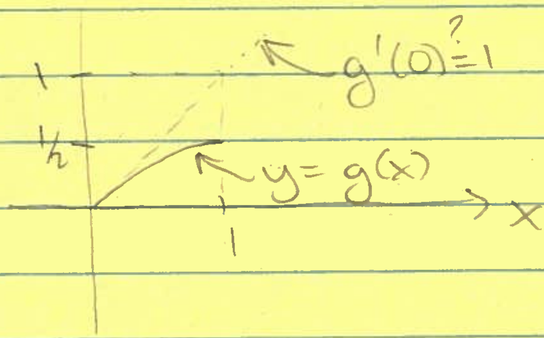
Notice that $g'(x)=0$ occurs @ x-values where $f(x)=0$. [$x=1,3$ in this example.] In fact:

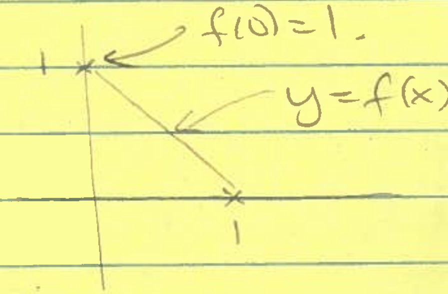
$$g'(x)>0 \iff f(x)>0$$

$$g'(x)=0 \iff f(x)=0$$

$$g'(x)<0 \iff f(x)<0$$

Moreover, there even appears to be quantitative agreement.





$$g'(0)\overset{?}{=}f(0).$$

In fact, this is the first part of the FTC:

<u>FTC1</u> If:

$$g(x)=\int_0^x f(t)\,dt$$

then:

$$g'(x)=f(x).$$

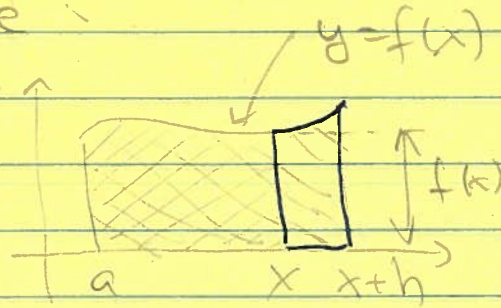
(See text for more precise statement.)

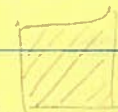
Sketch of proof:

Recall:

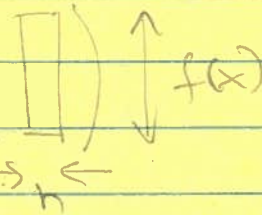
$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

Consider therefore



$g(x+h) - g(x) = $ area() - area()

$=$ area()

$\approx$ area()

$= hf(x)$

ie.

$$\frac{g(x+h) - g(x)}{h} \approx f(x)$$

The smaller h, the better we expect this approximation to be.

ie. $$\lim_{h \to 0} \frac{g(x+h)-g(x)}{h} = f(x)$$

or $$g'(x) = f(x)$$

$\boxtimes$

(See Text for more rigorous proof)

Another way to write FTC1 is:

$$\frac{d}{dx}\int_0^x f(t)\,dt = f(x). \qquad (1)$$

ie. integration (of f) followed by differentiation returns you back to f. Later, we will see that performing the operations in the reverse direction does the same thing (FTC2).

Ex $F(x) = \int_x^0 t\,dt$. Compute $F'(x)$.

Sol^n $$F(x) = -\int_0^x t\,dt \Rightarrow \frac{d}{dx}F(x) = -x \text{ by FTC1.}$$

Ex $h(x) = \int_1^{e^x} \ln t\,dt$. Compute $h'(x)$.

Sol^n Let $u = e^x$. Then

$$\frac{d}{dx}\int_1^{e^x} \ln t\,dt = \frac{d}{dx}\int_1^u \ln t\,dt$$

$$= \underbrace{\frac{d}{du}\int^{u} \ln t \, dt}_{\ln u \;(\text{FTC1})} \cdot \underbrace{\frac{du}{dx}}_{e^x \;(u=e^x)}$$

$$= \ln u \cdot e^x = x e^x$$

FTC2

$$\int_a^b f(x)\,dx = F(b) - F(a) \qquad (*)$$

where $F' = f$.

Pf Let

$$g(x) = \int_a^x f(t)\,dt$$

Then FTC1 $\Rightarrow$

$$g'(x) = f(x)$$

ie. g and F are antiderivatives of f and thus differ by a constant:

$$F(x) = g(x) + c.$$

Thus LHS of $(*)$ can be written

$$F(b) - F(a) = g(b) + c - [g(a) + c]$$

$$= g(b) - g(a)$$

$$= \int_a^b f(t)\,dt - \underbrace{\int_a^a f(t)\,dt}_{0}$$

$\boxtimes$

Note that since $F' = f$, we can also state FTC2 as:

$$\int_a^b F'(x)\,dx = F(b) - F(a). \qquad (2)$$

ie. first differtiating (F) and then integrating takes you back to where you were (F), but in the form $F(b) - F(a)$.

(1) and (2) together say that differentiatn and integration are universe processes.

EX Compute

$$\int_0^4 2^s\,ds$$

Sol^n Recall from L1 (p3) that:

$$\frac{d}{ds}\frac{1}{\ln 2}2^s = 2^s$$

Thus

$$\int_0^4 2^s\,ds = \frac{1}{\ln 2}2^s\Big|_0^4 = \frac{1}{\ln 2}(2^4 - 2^0)$$

$$= \frac{15}{\ln 2}$$

EX Compute $F'(x)$ where:

$$F(x) = \int_x^{x^2} e^{t^2}\,dt.$$

Sol^n

$$F(x) = \int_x^0 e^{t^2}\,dt + \int_0^{x^2} e^{t^2}\,dt.$$

$$= -\int_0^x e^{t^2}\,dt + \int_0^{x^2} e^{t^2}\,dt.$$

$$\Rightarrow \quad F'(x) = -e^{-x^2} + \underbrace{\frac{d}{dx}\int_0^{x^2} e^{t^2}\,dt}$$

$$\downarrow u = x^2$$

$$\frac{d}{du}\int_0^{u} e^{t^2}\,dt \cdot \frac{du}{dx}$$

$$= e^{u^2} \cdot 2x$$

$$= 2xe^{(x^2)^2}$$

Thus

$$F'(x) = -e^{-x^2} + 2xe^{x^4}$$