2B

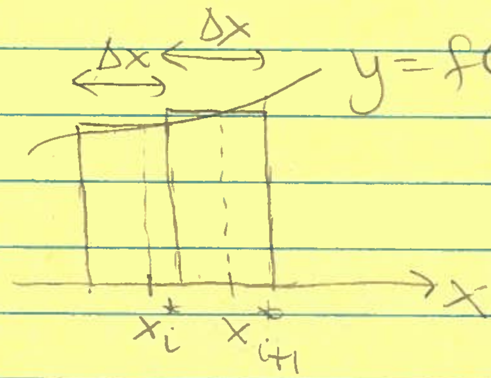
L3
§5.3

The Definite Integral.

Recall that we used the limit of a large # rectangles to define what we mean by the area under a curve. We now formalize this notion and use a special symbol, $\int$, to denote the limit:

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i^*)\,\Delta x$$

where



MIDPOINT RULE
Choosing x_i^* to be midpoint of base ("midpoint rule") is usually better than left or right edge

Note: x_i^* can be anywhere in the base of rectangle i.

Note:

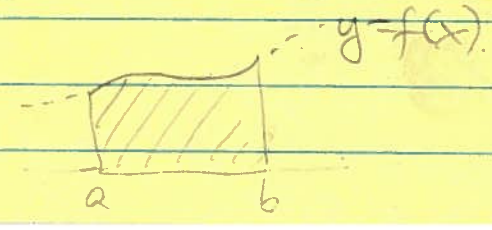
$$\int_a^{b} f(x)\,dx$$

$b \leftarrow$ "upper limit of integration"

$f(x)$: "integrand"

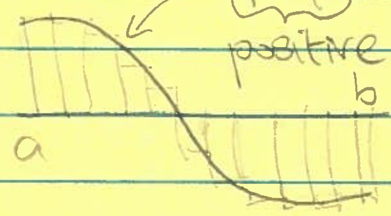
$$\sum_{i=1}^{n} f(x_i^*)\,\Delta x = \text{"Riemann Sum"}.$$

Note: $\int_a^b f(x)\,dx$ = area of:



... but what if $f(x)$ is not always positive? What is $\int_a^b f(x)\,dx$ then? No problem - just think of Riemann sums:

$\underbrace{f(x_i^*)}_{\text{positive}}\Delta x$ = area of rectangle



$\underbrace{f(x_i^*)}_{\text{negative}}\Delta x = -$(area of rect.)

Thus:

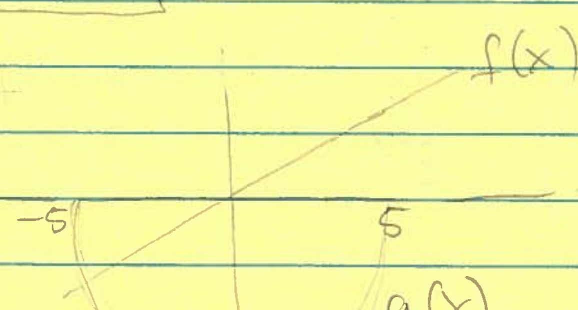
$$\int_a^b f(x)\,dx = A_1 - A_2$$

EX Evaluate:

$$I = \int_{-5}^{5} \left(x - \sqrt{25-x^2}\right)dx$$

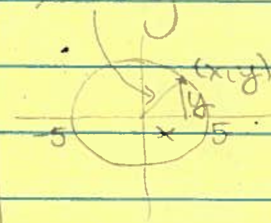
Sol^n First note that there are 2 functions here:

$\boxed{y = f(x) = x}$;



$y = g(x) = -\sqrt{25-x^2}$

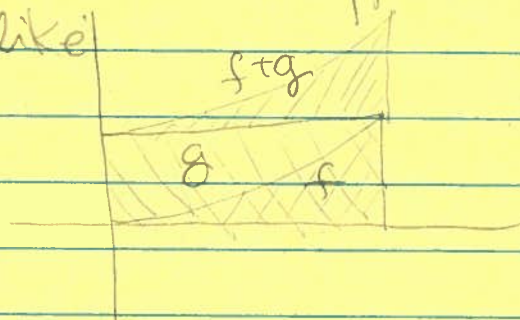
$x^2 + y^2 = 5^2$



Thus $$I = \int_{-5}^{5} [f(x) + g(x)]\,dx$$

$$= \int_{-5}^{5} f(x)\,dx + \int_{-5}^{5} g(x)\,dx$$

ASIDE: One way to see that this could be true is to suppose that f and g looked like



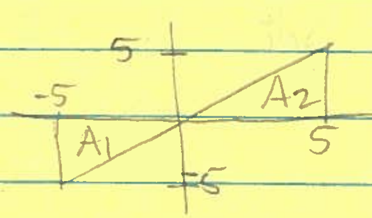
and to note that the area under f+g is the area under g plus area under f.

Returning to our problem:

$$I = \underbrace{\int_{-5}^{5} x\,dx}_{I_1} + \underbrace{\int_{-5}^{5} -\sqrt{25-x^2}\,dx}_{I_2}$$

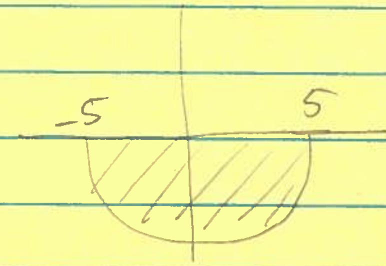
Now

$I_1 =$



$= -A_1 + A_2 = 0$ (symmetric)

and

$I_2 =$ 

$= -(\text{area of semi-disc})$

$= -\frac{1}{2}\pi(5)^2$

$= -\frac{25}{2}\pi.$

EX Given that

$$\int_0^{\pi} \sin^4 x\,dx = 3\pi/8,$$

What is $\displaystyle\int_{\pi}^{0} \sin^4\theta\,d\theta$?

Sol^n Think of the x as counting Riemann rectangles. We could have chosen to call this counter by any other name, eg. θ. Thus:

$$\int_0^{\pi} \sin^4 x\,dx = \int_0^{\pi} \sin^4\theta\,d\theta.$$

What about the order of the limits of integration? Notice that if we reverse 0 and π, then Δx in the

Riemann sum changes from $(\pi - 0)/n$ to $(0-\pi)/n$. Since this is the only change in the Riemann sum, we have:

$$\int_0^{\pi} \sin^4\theta \, d\theta = -\int_{\pi}^{0} \sin^4\theta \, d\theta .$$

Thus the answer to the question is $-3\pi/8$.

Ex Compute $\int_0^1 x\,dx$ using Riemann sums.

Sol'n

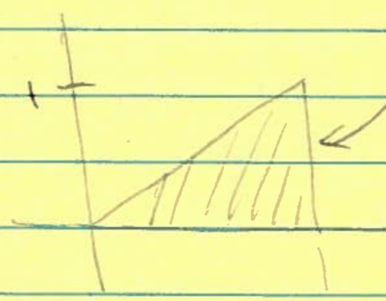
$$\int_0^1 x\,dx = \lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} i/n$$

$$= \lim_{n\to\infty} \frac{1}{n^2}\sum_{i=1}^{n} i$$

$$= \lim_{n\to\infty} \frac{1}{n^2}\,\frac{n(n+1)}{2}$$

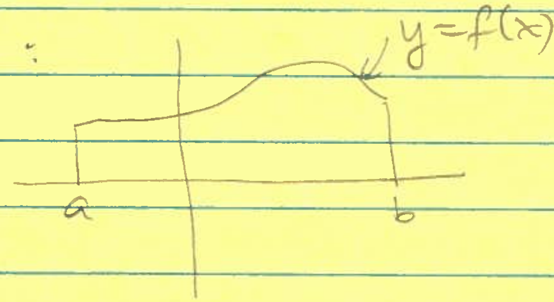
$$= \lim_{n\to\infty} \frac{1}{2}\left(1 + 1/n\right) = \frac{1}{2}$$

CHK:



area of Δ is $\frac{1}{2}(1)(1) = 1/2$.

Suppose $f(x) \geq 0$:



Then clearly :

$$\int_a^b f(x)\,dx \geq 0.$$

since the LHS is just an area, and areas are positive.

EX: Verify that

$$\int_0^4 (x^2-4x+4)\,dx \geq 0$$

without evaluating the integral.

Sol^n

$$\int_0^4 (x^2-4x+4)\,dx = \int_0^4 (x-2)^2\,dx$$

But

$$(x-2)^2 \geq 0$$

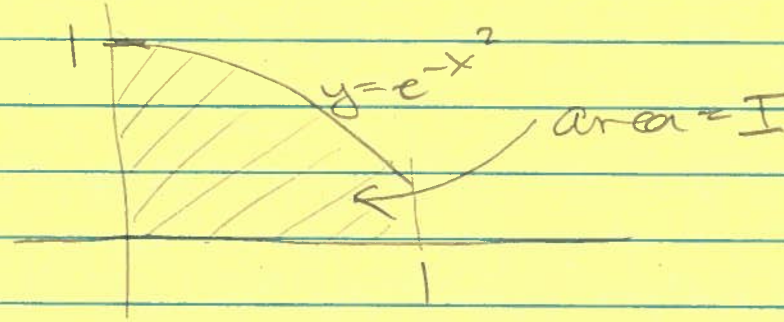
$$\Rightarrow \int_0^4 (x-2)^2\,dx \geq 0$$

$\boxtimes$

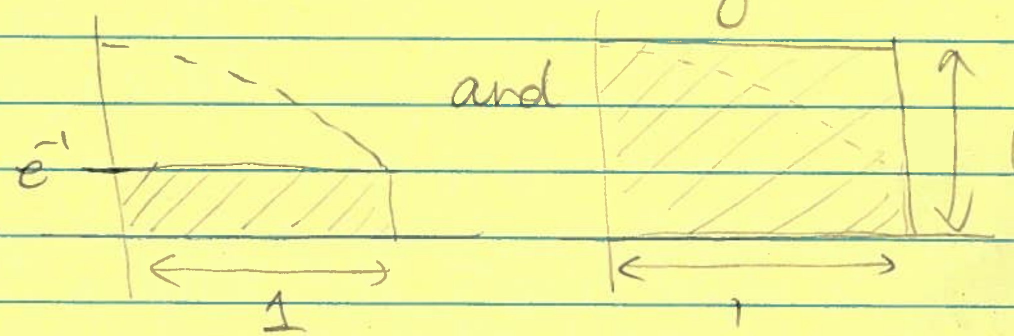
Ex Estimate

$$I = \int_0^1 e^{-x^2}\,dx.$$

Sol^n



Area above is bounded by



ie.

$$e^{-1}\cdot 1 \le I \le 1\cdot 1.$$

ie.
$$e^{-1} \le \int_0^1 e^{-x^2}\,dx \le 1.$$