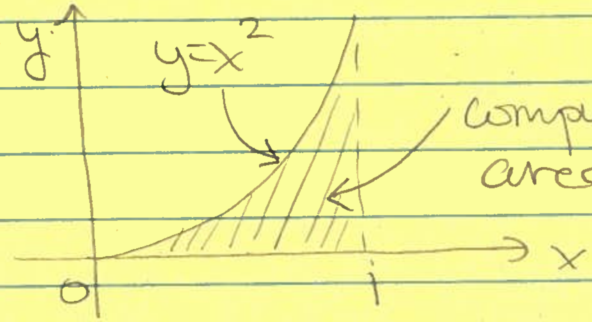
2B

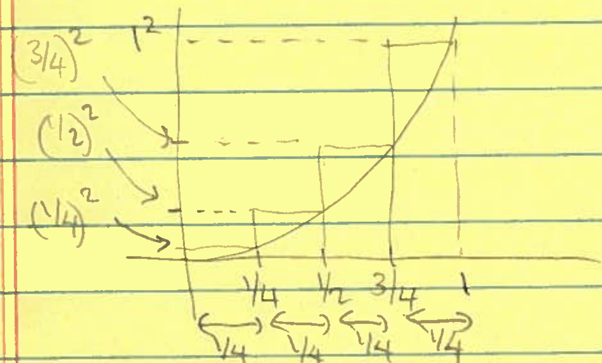
L2 §5.1

Areas and Distances.

How would you estimate the area under the parabola $y = x^2$ from 0 to 1



One approach is to divide the area into vertical strips:



and compute the sum of the areas of the rectangles:

$$\frac{1}{4}\left(\frac{1}{4}\right)^2 + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \frac{1}{4}\left(\frac{3}{4}\right)^2 + \frac{1}{4}(1)^2$$

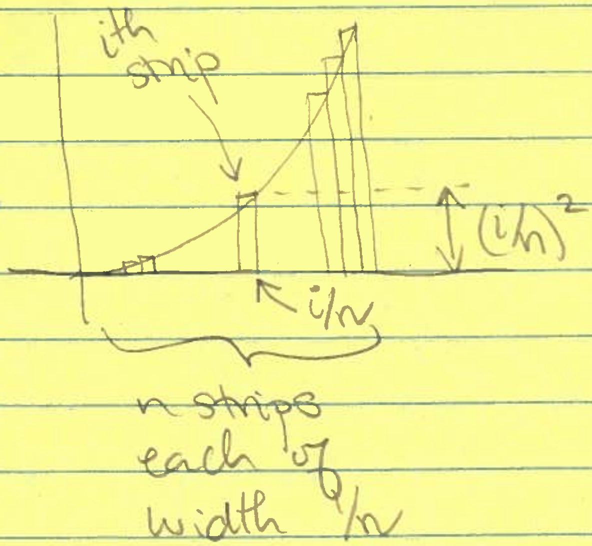
$$= 0.46875$$

This is an upper bound on the area because each strip overestimates the corresponding area under the curve

What if we wanted a better approximation? How could we do it?

Well, let's see what happens if we increase the number of strips while also decreasing their width (show animation). The set of strips collectively approximate the area under the curve better and better!

Let's pursue this by first generalizing the calculation above for an arbitrary number of strips, n, and then computing the limit, as $n \to \infty$, of the resulting expression.



Let R_n be the sum of the areas of the n strips.

Then $$R_n = \frac{1}{n}\left(\frac{1}{n}\right)^2 + \frac{1}{n}\left(\frac{2}{n}\right)^2 + \cdots + \frac{1}{n}\left(\frac{n}{n}\right)^2$$

$$= \underbrace{\frac{1}{n}\cdot\frac{1}{n^2}}_{1/n^{3}} \underbrace{\left(1^2 + 2^2 + \cdots + n^2\right)}_{\frac{n(n+1)(2n+1)}{6}}$$

← cf Ex5 App E of TEXT.

$$= \frac{(n+1)(2n+1)}{6n^2}$$

Now that we have a formula for R_n, lets carry out the second step of our program, and compute

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty} \frac{(n+1)(2n+1)}{6n^2}$$

$$= \frac{1}{6} \lim \frac{n+1}{n} \cdot \frac{2n+1}{n}$$

$$= \frac{1}{6} \lim \left(1+\frac{1}{n}\right)\left(2+1/n\right)$$

$$= \frac{1}{6} \cdot 1 \cdot 2 = 1/3$$

Since the strips clearly get better at covering the area under the curve as their number increases, we actually *define* the area under the curve to be $\lim\limits_{n\to\infty} R_n$.

JUST SO YOU KNOW:
Using simga notation, we may rewrite the sum more compactly as:

$$R_n = \sum_{i=1}^{n} \frac{1}{n}\left(\frac{i}{n}\right)^2 = \frac{1}{n^3}\sum_{i=1}^{n} i^2$$

$$= \frac{1}{n^3}\,\frac{n(n+1)(2n+1)}{6}$$

EX Compute the area under the curve $y = x^3$ from 0 to 1.

Sol$^{\underline{n}}$ Width of a strip is $\Delta x = 1/n$
Right endpoint of strip i is @ $x_i = i/n$

$$\therefore \text{area} = \lim_{n\to\infty} \sum_{i=1}^{n} \Delta x\, f(x_i)$$

$$= \lim \sum \frac{1}{n}\left(\frac{i}{n}\right)^3$$

$$= \lim_{n\to\infty} \frac{1}{n^4} \underbrace{\sum_{i=1}^{n} i^3}_{\left[\frac{n(n+1)}{2}\right]^2}$$

$$= \lim_{n\to\infty} \frac{1}{4}\frac{(n+1)^2}{n^2} = \lim_{n\to\infty} \frac{1}{4}\left(1+1/n\right)^2$$

$$= 1/4$$

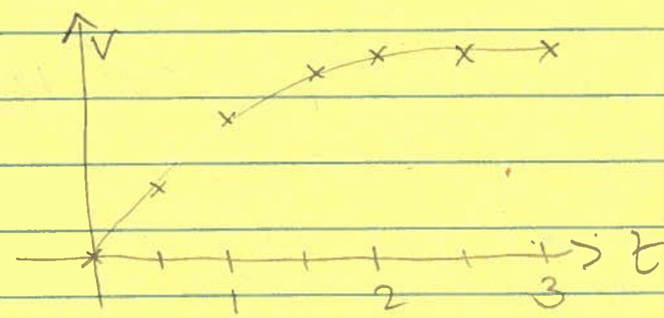
The method we developed above can be used to compute the distance traveled by an object during a certain time period if the velocity of the object is known @ all times.

EX: Data for runner in a race:

time, t (s)	0	0.5	1.0	1.5	2.0	2.5	3.0
speed, v (ft/s)	0	6.2	10.8	14.9	18.1	19.4	20.2

Estimate the distance traveled.

Sol^{n} Picture:

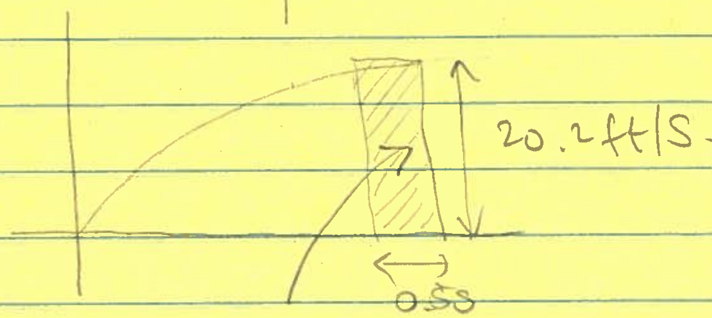


How far does runner travel between 2.5 and 3s? Speed is almost constant during that time (19.4 - 20.2 ft/s). So

$$\text{distance} = \text{speed} \times \text{time}$$

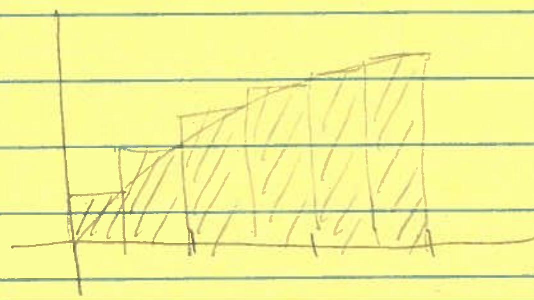
$$\text{ft} \qquad \frac{\text{ft}}{\cancel{\text{s}}} \times \cancel{\text{s}}$$

But wait a minute, speed × time is just width × height for a vertical strip!



area = distance traveled during 0.5 s interval

Could apply same reasoning to the previous 0.5s time intervals too to compute the distance traveled during those intervals:



Since the sum of distances traveled is the sum of the areas of strips, we have shown that an approx for the total distance traveled is:

$$0.5\,(6.2 + 10.8 + 14.9 + 18.1 + 19.4 + 20.2)$$

$$= 44.8\,ft.$$