L16
§11.1

Sequences.

A sequence is an $\boxed{\text{ordered list}}$ of numbers, e.g.

$$1, 2, 5, 4, 10, \ldots$$

The numbers are often called "terms".

In general a sequence is written

$$a_1, a_2, \ldots, a_n, \ldots$$

or $$\{a_n\}$$

or $$\{a_n\}_{n=1}^{\infty}$$

Ex Find a formula for a_n if the first few terms of the sequence are

$$\frac{3}{5}, -\frac{4}{25}, \frac{5}{125}, -\frac{6}{625}, \frac{7}{3125}, \ldots$$

Sol^n We have that:

	n	num	den	sgn
$a_1 = \frac{3}{5}$	1	1+2	5^1	$(-1)^0$
$a_2 = -\frac{4}{25}$	2	2+2	5^2	$(-1)^1$
$a_3 = \frac{5}{125}$	3	3+2	5^3	$(-1)^2$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$a_n = (-1)^{n-1}\frac{n+2}{5^n} \Leftarrow$	n	n+2	5^n	$(-1)^{n-1}$

Ex List first 5 terms of the sequence

$$a_1 = 2, \; a_2 = 1, \; a_{n+1} = a_n - a_{n-1}$$

Sol^n:

$a_1 = 2$

$a_2 = 1$

$a_3 = a_2 - a_1 = 1 - 2 = -1$

$a_4 = a_3 - a_2 = -1 - 1 = -2$

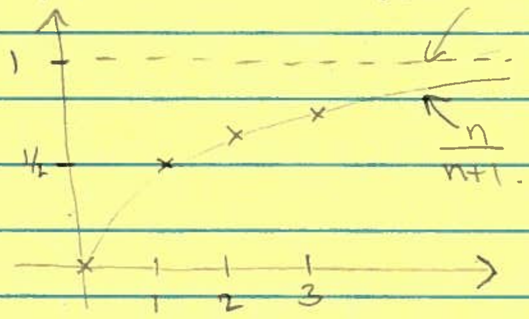
$a_5 = a_4 - a_3 = -2 - (-1) = -1.$

☒

EX Find $\lim\limits_{n\to\infty} \frac{n}{n+1}$

Sol^{n} Graph:

is 1 the limiting value of $\frac{n}{n+1}$?



$$\lim_{n\to\infty} \frac{n}{n+1} = \lim_{n\to\infty} \frac{1}{1+1/n} = \frac{\lim 1}{1+\lim 1/n} = \frac{1}{1+0} =$$

☒

EX Find $\lim\limits_{n\to\infty} \frac{\ln n}{n}$

Sol^{n} Here it is useful to think of $\frac{\ln n}{n}$ as $f(n)$ where

$$f(x) = \frac{\ln x}{x}$$

and x is real (ie not an integer). Then, by l'Hôpital:

$$\lim_{x\to\infty} \frac{\ln x}{x} = \lim_{x\to\infty} \frac{1/x}{1} = 0.$$

Since $f(x) \to 0$, the sequence $f(n) = \frac{\ln n}{n}$ "converges" to zero.

☒

(*) EX Find $\lim_{n\to\infty} \frac{(\ln n)^2}{n}$

Sol^n Again view $\frac{(\ln n)^2}{n}$ as $f(n)$ where $f(x) = \frac{(\ln x)^2}{x}$ and x is real. Then

$$\lim_{x\to\infty} \frac{(\ln x)^2}{x} \overset{H}{=} \lim \frac{2\ln x \cdot \frac{1}{x}}{1}$$

$$= 2 \lim \frac{\ln x}{x}$$

$$\overset{H}{=} 2 \lim \frac{1/x}{1}$$

$$= 0.$$

$f(x) \to 0 \Rightarrow f(n) \to 0.$

☒

EX Does $a_n = \frac{n}{\sqrt{a+n}}$ converge?

Sol^n

$$\lim \frac{\overset{\leftarrow \text{make this "1"}}{n}}{\sqrt{a+n}} = \lim \frac{1}{\sqrt{a+n}/n} = \frac{1}{\lim \frac{\sqrt{a+n}}{n}}$$

Now:

$$\frac{\sqrt{a+n}}{n} = \frac{\sqrt{a+n}}{\sqrt{n^2}} = \sqrt{\frac{a+n}{n^2}} = \sqrt{\frac{a}{n^2}+\frac{1}{n}}$$

Thus:

$$\lim \frac{\sqrt{a+n}}{n} = \sqrt{a \lim \frac{1}{n^2} + \lim \frac{1}{n}}$$

$$= \sqrt{a \cdot 0 + 0} = 0.$$

Thus:

$$\lim \frac{n}{\sqrt{a+n}} = \frac{1}{0} = \infty.$$

Since the limit of $\frac{n}{\sqrt{a+n}}$ is not finite, we say it "diverges".

▨

A sequence doesn't need to approach ∞ to be divergent:

EX What is $\lim_{n\to\infty} (-1)^n$?

Sol^n Sequence is $\underset{n=1}{-1}, \underset{n=2}{1}, \underset{n=3}{-1}, \underset{n=4}{1}, \ldots$

Clearly $(-1)^n$ does not approach any single number, ie. $\lim (-1)^n$ "does not exist". In this case, too, we say that the sequence "diverges".

▨

EX Find $\lim_{n\to\infty} \frac{n!}{n^n}$

Sol'n

$$\frac{n!}{n^n} = \frac{n\cdot(n-1)\cdot(n-2)\cdots 3\cdot 2\cdot 1}{n\cdot n\cdot n\cdots n\cdot n\cdot n}$$

$$= \underbrace{\frac{n\cdot(n-1)\cdots 2}{n\cdot n\cdots n}}_{\leq 1}\cdot\frac{1}{n}$$

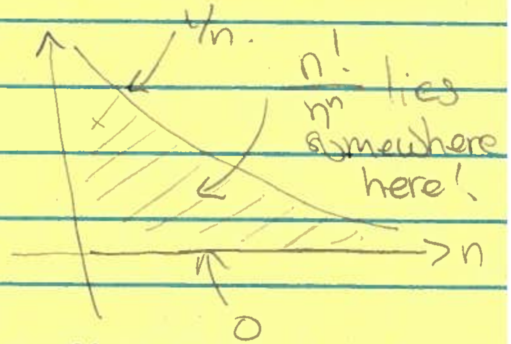
$$\leq \frac{1}{n}.$$

Also:

$$0 < \frac{n!}{n^n}$$

Thus:

$$0 < \frac{n!}{n^n} < \frac{1}{n}.$$



"Squeeze theorem".

Thus:

$$\lim_{n\to\infty}\frac{n!}{n^n} = 0.$$

▨

EX Find $\lim_{n\to\infty} \frac{(-1)^n}{\sqrt{n}}$.

Sol^n

Note:

$$-|a_n| \le a_n \le |a_n|$$

Thus, if $|a_n| \to 0$, then $-|a_n| \to 0$, and so does a_n by "squeeze theorem."

In our case:

$$|a_n| = \frac{1}{\sqrt{n}} \to 0$$

Thus

$$\frac{(-1)^n}{\sqrt{n}} \to 0$$

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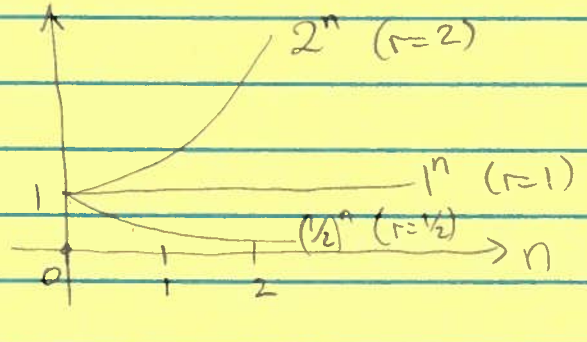
What is the behavior of r^n as $n \to \infty$? :



EX Find $\lim\limits_{n\to\infty} \dfrac{4^n}{1+9^n}$

Sol

$$\frac{4^n}{1+9^n} = \frac{4^n/9^n}{1/9^n+1} = \frac{(4/9)^n}{(1/9)^n+1} \longrightarrow \frac{0}{0+1}$$

since $r^n \to 0$ when $|r|<1$.

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(*) **EX** Find $\lim\limits_{n\to\infty} \dfrac{n^2}{\sqrt{n^3+4n}}$

Sol

$$\frac{n^2}{\sqrt{n^3+4n}} = \frac{n^2/n^{3/2}}{\sqrt{1+4/n^2}} = \frac{\sqrt{n}}{\sqrt{1+4/n^2}} \longrightarrow \frac{\infty}{\sqrt{1+0}} = \infty$$

make this "1".

☑

Consider the sequence $a_n = \dfrac{1-n}{2+n}$.

Is $a_n > a_{n+1}$?

This is equivalent to asking:

$$\frac{1-n}{2+n} \overset{?}{>} \frac{1-(n+1)}{2+(n+1)}$$

$$\iff (1-n)(n+3) \overset{?}{>} -(2+n)n$$

$$\iff \underset{-3n}{n+3} \; -n^2 \overset{?}{>} -2n-n^2$$

$$\iff -n^2 - 2n + 3 \overset{?}{>} -n^2 - 2n$$

$$\iff 3 \overset{?}{>} 0 .$$

which is true. We say that an is "decreasing".

Conversely a seq for which $a_n < a_{n+1}$ is said to be "increasing".

A seq is "monotonic" if it is either increasing or decreasing.

Moreover the seq $\frac{1-n}{2+n}$ is "bounded" between $a_1 = \underset{\substack{\uparrow \\ \text{upper bound}}}{0}$ and $a_\infty = \underset{\substack{\uparrow \\ \text{lower bound.}}}{-1}$.