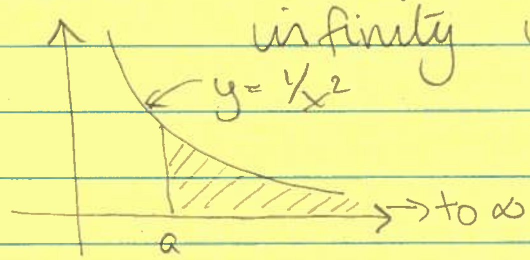
L15
§7.8

Improper Integrals.

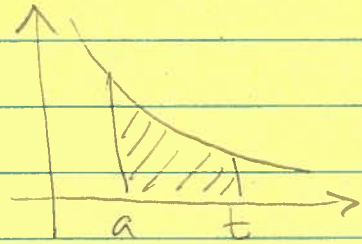
Consider the shaded region that extends to infinity in one direction.



Q. Is the area of the shaded region infinite?
A. No!

Here's why:

Consider a sub-region:



It's area is:

$$A(t) = \int_a^t \frac{1}{x^2}\,dx$$

$$= \left[-\frac{1}{x}\right]_a^t = \frac{1}{a} - \frac{1}{t}$$

By sending $t \to \infty$, the sub-region becomes the original region, whose area is therefore:

$$\lim_{t\to\infty} A(t) = \lim_{t\to\infty} \left[\frac{1}{a} - \frac{1}{t}\right]$$

$$= \frac{1}{a} - \lim_{t\to\infty} \frac{1}{t}$$

$$= \frac{1}{a} < \infty.$$

Def$^{\underline{n}}$ $\quad \int_a^\infty f(x)\,dx := \lim_{t\to\infty} \int_a^t f(x)\,dx.$

- LHS is called an improper integral.
- If RHS exists, then improper integral is called convergent; otherwise it is divergent.

EX Is $\int_1^\infty \frac{1}{x}\,dx$ convergent or divergent?

Sol$^{\underline{n}}$ $\quad \lim_{t\to\infty} \int_1^t \frac{1}{x}\,dx = \lim_{t\to\infty} [\ln t]_1^t = \lim_{t\to\infty} \ln t = \infty.$

$\Rightarrow \int_1^\infty \frac{1}{x}\,dx$ is divergent. ☒

Comparing the first and second examples, we see that $1/x^2$ goes to zero fast enough as $x \to \infty$ that the growth of the sub-region's area arrests.

EX Evaluate $\int_{-\infty}^0 x e^x\,dx$

Sol$^{\underline{n}}$ $\quad \int_{-\infty}^0 x e^x\,dx := \lim_{t\to-\infty} \int_t^0 x e^x\,dx$

Evaluate the second integral by parts:

$u = x \quad ; \quad dv = e^x\,dx$

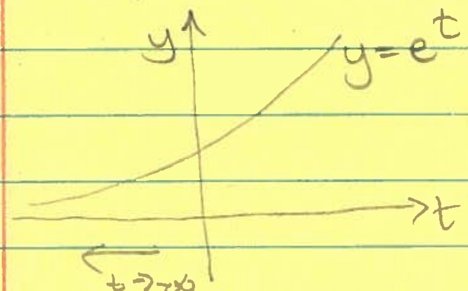
$du = dx \quad ; \quad v = e^x$

$$\int_t^0 xe^x\,dx = \left[xe^x\right]_t^0 - \int_t^0 e^x\,dx = \left[xe^x - e^x\right]_t^0$$

$$= [-1] - [te^t - e^t]$$

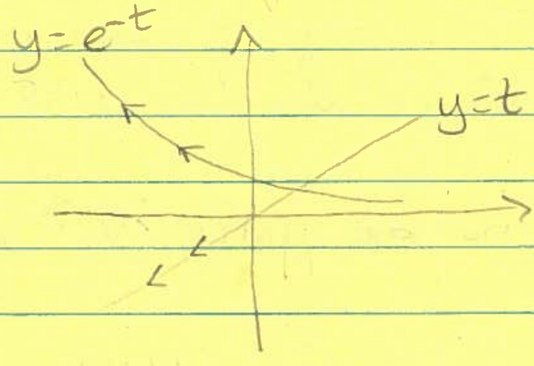
$$= -te^t + e^t - 1$$

We must evaluate the limit of this expression as $t \to -\infty$. Obviously $\lim_{t\to-\infty} 1 = 1$, while:



$$\Rightarrow \lim_{t\to-\infty} e^t = 0.$$

But what about $\lim_{t\to-\infty} te^t = \lim_{t\to-\infty} \frac{t}{e^{-t}}$?



Both t and e^{-t} grow in magnitude, but exponential growth is faster, making t/e^{-t} tend to 0.

Pf: Recall l'Hôpital's Rule:

$$\lim_{x\to\infty} \frac{f(x)}{g(x)} = \lim_{x\to\infty} \frac{f'(x)}{g'(x)}$$

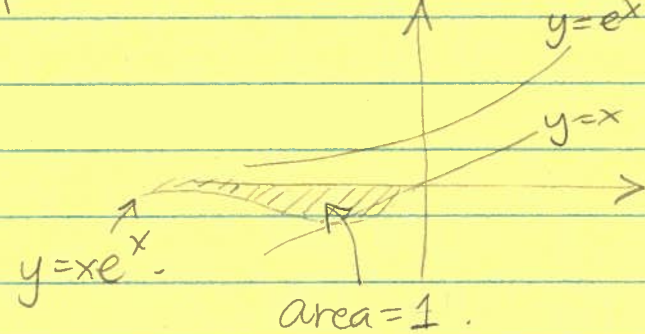
$$\Rightarrow \lim_{t\to-\infty} \frac{t}{e^{-t}} = \lim_{t\to-\infty} \frac{1}{e^{-t}(-1)} = 0$$

Thus:

$$\lim_{t\to-\infty}\int_t^0 xe^x\,dx = -1$$

ie.

$$\int_{-\infty}^0 xe^x\,dx = -1$$



EX Evaluate $\int_{-\infty}^{\infty} xe^{-x^2}dx$.

Sol^n

$$\int_{-\infty}^{\infty} xe^{-x^2}dx = \int_{-\infty}^{0} xe^{-x^2}dx + \int_0^{\infty} xe^{-x^2}dx$$

- $\int xe^{-x^2}dx = \int e^{-x^2}\cdot x\,dx = \int e^{-u}\tfrac{1}{2}du$
  $= -\tfrac{1}{2}e^{-u} = -\tfrac{1}{2}e^{-x^2}$

- $\int_t^0 xe^{-x^2}dx = \left[-\tfrac{1}{2}e^{-x^2}\right]_t^0 = \left[-\tfrac{1}{2}\right]-\left[-\tfrac{1}{2}e^{-t^2}\right]$
  $= \tfrac{1}{2}e^{-t^2} - \tfrac{1}{2}$

$\Rightarrow \lim_{t\to-\infty}\int_t^0 xe^{-x^2}dx = -\tfrac{1}{2} = \int_{-\infty}^0 xe^{-x^2}dx$

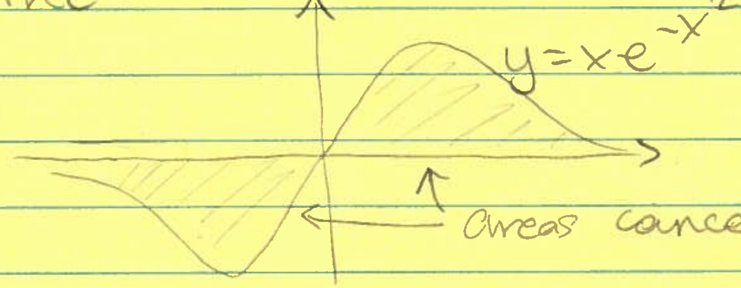
- $\int_0^t xe^{-x^2}dx = \left[-\tfrac{1}{2}e^{-x^2}\right]_0^t = \left[-\tfrac{1}{2}e^{-t^2}\right]-\left[-\tfrac{1}{2}\right]$

$\Rightarrow \lim_{t\to\infty}\int_0^t xe^{-x^2}dx = \tfrac{1}{2} = \int_0^{\infty} xe^{-x^2}dx$

Thus:

$$\int_{-\infty}^{\infty} x e^{-x^2} dx = \left(-\tfrac{1}{2}\right) + \left(\tfrac{1}{2}\right) = 0.$$

In hindsight, we would have gotten the answer by observing that the integrand is asymmetric:



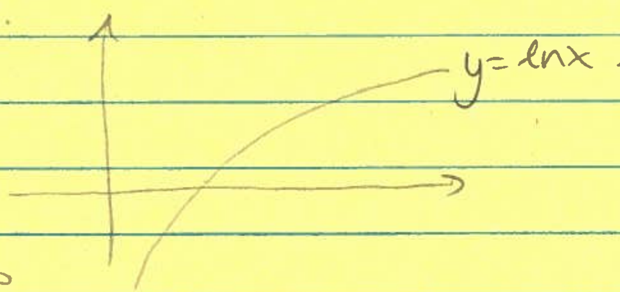
EX Is $\int_1^{\infty} \frac{\ln x}{x} dx$ convergent or divergent?

Sol^n $u = \ln x$
$du = \frac{1}{x} dx$

$$\int u\,du = \tfrac{1}{2}u^2$$

$$\Rightarrow \int_1^t \frac{\ln x}{x} dx = \tfrac{1}{2}\left[(\ln x)^2\right]_1^t = \tfrac{1}{2}(\ln t)^2$$

but:



$$\Rightarrow \lim_{t\to\infty} (\ln t)^2 = \infty$$

$\therefore \int_1^{\infty} \frac{\ln x}{x} dx$ is divergent.

(*) **EX** Compute $\int_1^\infty \frac{\ln x}{x^2}\,dx$.

Sol^n Evaluate $\int_1^t \frac{\ln x}{x^2}\,dx$ by parts:

$u = \ln x \qquad dv = \frac{1}{x^2}\,dx$

$du = \frac{1}{x}\,dx \qquad v = -\frac{1}{x}$

$$\int_1^t \frac{\ln x}{x^2}\,dx = \left[-\frac{1}{x}\ln x\right]_1^t - \int_1^t \left(-\frac{1}{x}\right)\cdot\frac{1}{x}\,dx$$

$$= \left[-\frac{\ln x}{x} + \frac{x^{-1}}{-1}\right]_1^t$$

$$= \left[-\frac{1}{x}(\ln x + 1)\right]_1^t$$

$$= \left[-\frac{1}{t}(\ln t + 1)\right] - \left[-1\right]$$

$$= -\frac{\ln t}{t} - \frac{1}{t} + 1.$$

Now: $\ln t$ and t both diverge as $t \to \infty$ so we try l'Hôpital's Rule:

$$\lim_{t\to\infty} \frac{\ln t}{t} = \lim_{t\to\infty} \frac{1/t}{1} = 0.$$

Thus:

$$\int_1^\infty \frac{\ln x}{x^2}\,dx = \lim_{t\to\infty} \int_1^t \frac{\ln x}{x^2}\,dx = 1.$$

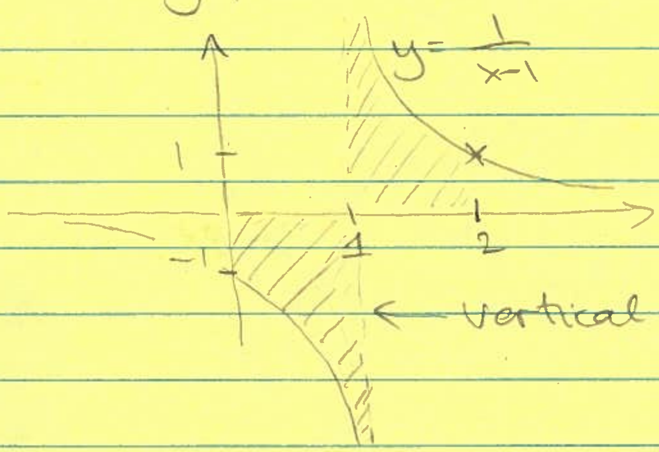
Consider: $\displaystyle\int_0^2 \frac{dx}{x-1}$

You might be tempted to write:

$$\int_0^2 \frac{dx}{x-1} = \Big[\ln|x-1|\Big]_0^2 = \ln 1 - \ln 1 = 0.$$

BUT THIS IS WRONG!

To see why, examine the integrand:



← vertical asymptote

We see that the shaded areas extend to ∞ along the vertical asymptote.

The vertical asymptote opens up the possibility that the shaded areas are each infinity. To find out, we must use limits:

$$\int_0^2 \frac{dx}{x-1} = \lim_{t\to 1^-} \underbrace{\int_0^t \frac{dx}{x-1}}_{\Big[\ln|x-1|\Big]_0^t = \ln|t-1|} + \lim_{t\to 1^+} \int_t^2 \frac{dx}{x-1}$$

But $t \to 1^- \Rightarrow |t-1| \to 0 \Rightarrow \ln|t-1| \to -\infty$

$\Rightarrow \int_0^1 \frac{dx}{x-1}$ is divergent.

This already implies that $\int_0^2 \frac{dx}{x-1}$ is divergent.

(*) **Ex** Is $\int_0^4 \frac{dx}{x^2-x-2}$ convergent or divergent?

Soln

$$\frac{1}{x^2-x-2} = \frac{1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2}$$

$$\Rightarrow \quad 1 = A(x-2) + B(x+1).$$

$x = 2 \Rightarrow 1 = B(3) \Rightarrow B = 1/3$

$x = -1 \Rightarrow 1 = A(-3) \Rightarrow A = -1/3.$

$$\frac{1}{x^2-x-2} = \frac{-1/3}{x+1} + \underbrace{\frac{1/3}{x-2}}_{\substack{\text{vertical asymptote} \\ \text{at } x=2}}$$

Thus:

$$\int_0^4 \frac{dx}{x^2-x-2} = \int_0^4 \frac{-\frac{1}{3}dx}{x+1} + \frac{1}{3}\lim_{t\to 2^-}\int_0^t \frac{dx}{x-2}$$

$$+ \frac{1}{3}\lim_{t\to 2^+}\int_t^4 \frac{dx}{x-2}$$

Consider:

$$\int_0^t \frac{dx}{x-2} = \Big[\ln|x-2|\Big]_0^t = \ln|t-2| - \ln 2$$

But: $t \to 2 \Rightarrow |t-2| \to 0 \Rightarrow \ln|t-2| \to -\infty$.

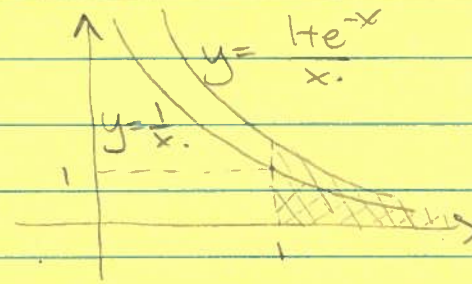
Thus $\int_0^4 \frac{dx}{x^2-x-2}$ is divergent.

COMPARISON TEST

Is $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ convergent or divergent?

A neat trick is to observe that

$$\frac{1+e^{-x}}{x} > \frac{1}{x}.$$



$$\Rightarrow \int_1^\infty \frac{1+e^{-x}}{x}\,dx > \int_1^\infty \frac{1}{x}\,dx$$

But $\int_1^\infty \frac{1}{x}\,dx$ diverges (p 2). Thus $\int_1^\infty \frac{1+e^{-x}}{x}\,dx$ must diverge too!