23 -1-

44 §7.5

Strategy for Integration

Integration is tough, so it's helpful to review strategy:

1. Simplify the integrand, if possible, eg

$$\int \frac{\tan\theta}{\sec^2\theta} d\theta = \int \frac{\sin\theta}{\cos\theta} \cdot \cos^2\theta \cdot d\theta.$$

$$= \int \sin\theta \cos\theta \, d\theta = \frac{1}{2}\int \sin(2\theta)\, d\theta$$

2. Look for obvious substitutions.

3. Classify integrand
   (a) trig functions
   (b) rational functions.
   (c) product of powers of x and transcendental functions (exp, sin, etc)
   (d) radicals

4. Ask whether more than one of the previous steps need to be performed.

P3L5

$\boxed{\text{Memorize, atleast: 7, 9, 11, 13, 17, 18 p. 503 of TEXT}}$

verify by differentiating RHS after rewriting in terms of cos and sin, eg p2L5

p10L12 p4L6

$\cos^2 x + \sin^2 x = 1$

$\sec^2 x = 1 + \tan^2 x.$

Ex $\int \frac{\cos x}{1-\sin x}\,dx$

Sol $u = 1-\sin x$
$du = -\cos x\,dx$

$\int \frac{-du}{u} = -\ln|u| + C.$
$= -\ln|1-\sin x| + C.$

Ex $\int \sqrt{y}\,\ln y\,dy$

Sol^n $u = \ln y$; $dv = \sqrt{y}\,dy$
$du = \frac{1}{y}dy$; $v = \frac{y^{3/2}}{3/2}$

$$\int \sqrt{y}\ln y\,dy = \frac{2}{3}y^{3/2}\ln y - \underbrace{\int \frac{2}{3}y^{3/2}\cdot\frac{1}{y}dy}$$

$\frac{2}{3}\int y^{1/2}dy$
$= \frac{2}{3}\frac{y^{3/2}}{3/2} + C.$

Thus
$$\int \sqrt{y}\ln y\,dy = \frac{2}{3}y^{3/2}\ln y - \left(\frac{2}{3}\right)^2 y^{3/2} + C.$$

① Ex $\int \frac{t\,dt}{t^4+2}$

Sol $u = t^2$
$du = 2t\,dt$

$\int \frac{t\,dt}{t^4+2} = \int \frac{\frac{1}{2}du}{u^2+2} = \frac{1}{2}\int \frac{du}{u^2+2}$

Recall: $\boxed{\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)}$

Thus:

$$\int \frac{t\,dt}{t^4+2} = \frac{1}{2}\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{t^2}{\sqrt{2}}\right) + C.$$

EX $\int \tan x\,dx$.

SOL $\int \frac{\sin x}{\cos x}\,dx$; $u = \cos x$.
$du = -\sin x\,dx$.

$$\int \frac{\sin x\,dx}{\cos x} = \int \frac{-du}{u} = -\ln|u| + c.$$
$$= -\ln|\cos x| + c.$$
$$= \ln\left|\frac{1}{\cos x}\right| + c.$$
$$= \ln|\sec x| + c.$$

ie. $\boxed{\int \tan x\,dx = \ln|\sec x| + c}$

② EX $\int \frac{\cos(1/x)}{x^3}\,dx$

SOL Write: $\int \frac{1}{x}\cdot\frac{\cos(1/x)}{x^2}\,dx$.

$u = \frac{1}{x}$; $dv = \frac{\cos(1/x)}{x^2}\,dx$

$du = -\frac{dx}{x^2}$; $v = \int \frac{\cos(1/x)}{x^2}\,dx$.

Evaluate v by substitution!

$w = 1/x \quad dw = -dx/x^2$.

$\Rightarrow v = -\int \cos w \, dw = -\sin w = -\sin(1/x)$.

Thus:

$$\int \frac{1}{x} \cdot \frac{\cos(1/x)}{x^2} dx = -\frac{1}{x} \cdot \sin\left(\frac{1}{x}\right) - \underbrace{\int \sin\left(\frac{1}{x}\right) \cdot \frac{dx}{x^2}}_{I}$$

I can be evaluated by substitution.

$w = 1/x \quad dw = -dx/x^2$

$I = -\int \sin w \, dw = \cos w = \cos(1/x)$

$$\therefore \int \frac{1}{x} \cdot \frac{\cos(1/x)}{x^2} dx = -\frac{\sin(1/x)}{x} - \cos(1/x) + C.$$

---

③ Ex $\displaystyle \int \frac{1}{x^3\sqrt{x^2-1}} dx$

Sol^n Recall $\sec^2\theta = 1 + \tan^2\theta$

or $\sec^2\theta - 1 = \tan^2\theta$

This suggests the trig substitution $x = \sec\theta$

$\Rightarrow \sqrt{x^2-1} = |\tan\theta|$

$= \tan\theta$

... if θ lies in these ranges:

$dx = \sec\theta\tan\theta\, d\theta$.

(chk this by writing $\sec\theta$ as $1/\cos\theta$).

Thus:

$$\int \frac{dx}{x^3\sqrt{x^2-1}} = \int \frac{\sec\theta\tan\theta\, d\theta}{\sec^3\theta\tan\theta}$$

$$= \int \cos^2\theta\, d\theta$$

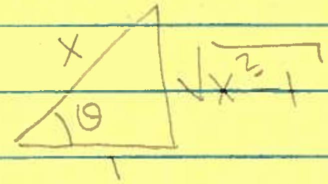
$$= \frac{1}{2}\int [1+\cos(2\theta)]\, d\theta$$

$$= \frac{1}{2}\theta + \frac{1}{2}\frac{1}{2}\cdot\sin(2\theta) + C$$

$$= \frac{1}{2}\theta + \frac{1}{2}\cos\theta\sin\theta + C.$$

We can write $\theta = \sec^{-1}x$ but what are $\sin\theta$ and $\cos\theta$? Well, $\sec\theta = x \Rightarrow \cos\theta = 1/x$

$\Rightarrow$



$\Rightarrow \sin\theta = \frac{\sqrt{x^2-1}}{x} = \sqrt{1-\frac{1}{x^2}}$

$\Rightarrow \quad \cos\theta \sin\theta = \dfrac{\sqrt{x^2-1}}{x^2}$

EX $\quad \int \ln(1+x^2)\,dx$

Soln $\quad u = \ln(1+x^2) \quad ; \quad dv = dx$

$du = \dfrac{1}{1+x^2}\cdot 2x\,dx \quad ; \quad v = x.$

$$\int \ln(1+x^2)\,dx = x\ln(1+x^2) - 2\int \underbrace{\frac{x^2}{1+x^2}}\,dx$$

$$\begin{array}{r} 1 \\ 1+x^2 \overline{)\, x^2 } \\ \underline{x^2+1} \\ -1 \end{array}$$

$\Leftarrow$ rational function ; can we divide $1+x^2$ into x^2 to get simpler rational function?

$\Rightarrow \quad x^2 = (1+x^2)\cdot 1 - 1 \quad \Rightarrow \quad \dfrac{x^2}{1+x^2} = 1 - \underbrace{\dfrac{1}{1+x^2}}_{\text{irreducible quadratic function}}.$

Thus:
$$\int \frac{x^2\,dx}{1+x^2} = \int dx - \int \frac{dx}{1+x^2}$$

$$= x - \tan^{-1}x + C$$

and

$$\int \ln(1+x^2)\,dx = x\ln(1+x^2) - 2x + 2\tan^{-1}x + C$$

---

④ **Ex** $\int e^{x+e^x}\,dx$

Sol^n $\int e^{x+e^x}\,dx = \int e^{e^x}\cdot e^x\,dx$ $u = e^x$, $du = e^x\,dx$

$= \int e^u\,du$

$= e^u + c = e^{e^x} + C.$

⑤ **Ex** $\int \arctan\sqrt{x}\,dx$

Sol^n $t = \sqrt{x} \Rightarrow t^2 = x \Rightarrow dx = 2t\,dt$

$\int \arctan\sqrt{x}\,dx = \int \arctan t \cdot 2t\,dt.$

$u = \arctan t$	$dv = 2t\,dt$
$du = \dfrac{1}{1+t^2}\,dt.$	$v = 2\cdot\dfrac{t^2}{2} = t^2$

$\int \arctan t \cdot 2t\,dt = t^2\arctan t - \underbrace{\int \frac{t^2\,dt}{1+t^2}}_{}$

$= t - \arctan t$ (p6)

Thus

$\int \arctan\sqrt{x}\,dx = x\arctan\sqrt{x} - \sqrt{x} + \arctan\sqrt{x} + C$

$= (x+1)\arctan\sqrt{x} - \sqrt{x} + C$

(b) **Ex** $\int \frac{dx}{1+e^x}$

Sol^n

$u = 1+e^x$
$du = e^x dx$
$= (u-1)dx$
$\Rightarrow dx = \frac{du}{u-1}$

$$\int \frac{dx}{1+e^x} = \int \frac{du}{u(u-1)}$$

Now:

$$\frac{1}{u(u-1)} = \frac{A}{u} + \frac{B}{u-1} \qquad (*)$$

Multiply $(*)$ across by $u(u-1)$:

$$1 = A(u-1) + Bu$$

$u=1$: $B=1$
$u=0$: $A=-1$

$$\Rightarrow \int \frac{dx}{1+e^x} = -\int \frac{du}{u} + \int \frac{du}{u-1}$$

$$= -\ln|u| + \ln|u-1| + C$$

$$= -\ln(1+e^x) + \ln e^x + C \qquad (**)$$

$$= x - \ln(1+e^x) + C.$$

Another Method:

$$\int \frac{dx}{1+e^x} = \int \frac{e^{-x}dx}{1+e^{-x}}$$

$u = 1+e^{-x}$
$du = -e^{-x}dx$

$$= \int \frac{-du}{u}$$

$$= -\ln(1+e^{-x}) + C.$$

This is the same as previous answer since $(**)$ can be written (omitting C for clarity):

$$-\ln\left(\frac{1+e^x}{e^x}\right) = -\ln(e^{-x}+1)$$