L13
§7.4

Integration by partial fractions

Observe:

$$\underbrace{\frac{2}{x-1} - \frac{1}{x+2}}_{\text{add rational functions}} = \frac{2(x+2)-(x-1)}{(x-1)(x+2)} \leftarrow \text{least common denominator}$$

$$= \frac{x+5}{x^2+x-2} \qquad (*)$$

Now consider the integral:

$$\int \frac{x+5}{x^2+x-2}\,dx$$

In light of $(*)$, you might be tempted to write:

$$\int \frac{x+5}{x^2+x-2}\,dx = \int \frac{2dx}{x-1} - \int \frac{dx}{x+2}$$

$$= 2\ln|x-1| - \ln|x+2| + C.$$

It turns out that it's possible to express any function $P(x)/Q(x)$ as a sum of "partial fractions", as above, provided the degree of the polynomial P is less than that of Q.

Recall: if:

$$P(x) = 5x^3 + 4x^2 + x + 2$$

then 3 is the degree of P.

What then are we to do with

$$\frac{x^3+x}{x-1} \begin{array}{l} \leftarrow \text{degree} = 3 \\ \leftarrow \text{degree} = 1. \end{array} \quad ?$$

Answer: perform long division to make the degree of the numerator smaller than that of the denominator:

$$\begin{array}{r|l} & x^2 + x + 2 \\ \hline x-1 & x^3 + 0x^2 + x \\ & \underline{x^3 - x^2} \qquad \downarrow \\ & x^2 + x \\ & \underline{x^2 - x} \\ & 2x \\ & \underline{2x - 2} \\ & 2 \end{array}$$

"polynomial long division"

Thus:

$$\underbrace{x^3+x}_{\text{dividend}} = \underbrace{(x-1)}_{\text{divisor}}\underbrace{(x^2+x+2)}_{\text{quotient}} + \underbrace{2}_{\text{remainder}}.$$

or

$$\frac{x^3+x}{x-1} = x^2+x+2 + \frac{2}{x-1}.$$

Again, this is useful:

$$\int \frac{x^3+x}{x-1}\,dx = \int (x^2+x+2)\,dx + 2\int \frac{dx}{x-1}$$

$$= \frac{x^3}{3} + \frac{x^2}{2} + 2x + 2\ln|x-1| + C.$$

In the last example, long division produced a partial fraction, $\frac{2}{x-1}$, that was as simple as it gets.

The next example shows what to do if the denominator of the partial fraction is of higher degree:

EX Evaluate
$$\int \frac{x^2+2x-1}{2x^3+3x^2-2x}\,dx$$

SOL: Since the degree of the denominator is larger than that of the numerator, we don't need to perform long division.

Thus we have a "rational function" (the integrand) that can be decomposed into a sum of partial fractions, similar to the example at the beginning of the lecture (cf. (*)).

Key to the decomposition in that example was the fact that:
$$x^2+x-2 = (x-1)(x+2)$$

In fact, any polynomial (whatever its degree) can be factored as a product of linear factors (degree 1) and irreducible quadratic factors (degree 2). Remarkable, but true!

In our case:

linear   quadratic, but reducible.

$$2x^3 + 3x^2 - 2x = x\,\underbrace{(2x^2+3x-2)}$$

$$= x\,(2x-1)(x+2) \quad (3 \text{ linear factors})$$

Following (*), we write:

$$\frac{x^2+2x-1}{x(2x-1)(x+2)} = \frac{A}{x} + \frac{B}{2x-1} + \frac{C}{x+2}$$

To determine A, B, C, multiply across by denominator of LHS:

$$x^2+2x-1 = A(2x-1)(x+2) + Bx(x+2) + Cx(2x-1)$$

Next write RHS in the standard form for polynomials:

$$x^2+2x-1 = (2A+B+2C)x^2$$
$$+(3A+2B-C)x$$
$$-2A$$

(**)

The polynomials on LHS and RHS are identical, so their coefficients (abbr. "coeffs") must be equal, yielding Eqns (1), (2), (3) below

ASIDE: To see this, note that LHS and RHS must match at all x, including $x=0$. Setting $x=0$ in (**) gives:

$$-1 = -2A \quad \text{(coeffs. of } x^0\text{)} \qquad (1)$$

Subtract (1) from both sides of (**):

$$x^2 + 2x = (2A+B+2C)x^2 + (3A+2B-C)x$$

Dividing across by x gives:

$$x+2 = (2A+B+2C)x + (3A+2B-C) \qquad (***)$$

Setting $x=0$ again:

$$2 = 3A + 2B - C. \quad \text{(coeffs of } x^1\text{)} \qquad (2)$$

Subtracting (2) from both sides of (***):

$$x = (2A+B+2C)x$$

Divide across by x:

$$1 = (2A+B+2C) \quad \text{(coeffs of } x^2\text{)} \qquad (3).$$

Now:

$$(1) \Rightarrow A = 1/2$$

$$(2),(3) \Rightarrow \begin{cases} 2B - C = \frac{1}{2} \\ B + 2C = 0 \end{cases}$$

$$\Rightarrow \quad \begin{array}{r} 2B - C = \frac{1}{2} \\ 2B + 4C = 0 \\ \hline -5C = \frac{1}{2} \end{array} \quad \Rightarrow C = -\frac{1}{10}$$

or

$$\begin{array}{r} 4B - 2C = 1 \\ B + 2C = 0 \\ \hline 5B \quad = 1 \end{array} \quad \Rightarrow B = 1/5.$$

So:

$$\int \frac{x^2+2x-1}{x(2x-1)(x+2)}\,dx = \int \frac{1/2\,dx}{x} + \int \frac{1/5\,dx}{2x-1} + \int \frac{-1/10\,dx}{x+2}$$

$$= \frac{1}{2}\ln|x| + \frac{1}{5}\cdot\frac{1}{2}\ln|2x-1|$$

$$- \frac{1}{10}\ln|x+2| + C.$$

Sometimes a linear factor is repeated in the decomposition of a polynomial. Here's what to do in that case:

Ex

$$\frac{1}{(t^2-1)^2} = \frac{1}{[(t+1)(t-1)]^2} = \frac{1}{(t+1)^2(t-1)^2}$$

$$= \underbrace{\frac{A}{t+1} + \frac{B}{(t+1)^2}}_{\text{one partial fraction for each power of } t+1 \text{ up to } 2} + \underbrace{\frac{C}{t-1} + \frac{D}{(t-1)^2}}_{\text{one partial fraction for each power of } t-1 \text{ up to } 2.}$$

Multiplying both sides by $(t+1)^2(t-1)^2$:

$$1 = A(t+1)(t-1)^2 + B(t-1)^2 + C(t+1)^2(t-1) + D(t+1)^2$$

$t=1 \Rightarrow$ $1 = D(1+1)^2 = 4D \Rightarrow D = 1/4.$

$t=-1 \Rightarrow$ $1 = B(-1-1)^2 = 4B \Rightarrow B = 1/4.$

$t=0 \Rightarrow$ $1 = A + B - C + D$

$\Rightarrow 1 = A + \frac{1}{4} - C + \frac{1}{4}.$

$$\Rightarrow A - C = \frac{1}{2} \qquad (4)$$

To find another equation, we resort to the earlier strategy of equating coefficients. Suppose we equate coeffs of t^3:

coeffs of t^3: $$0 = A + C \qquad (5)$$

(4), (5) $\Rightarrow$

$$\begin{aligned} A - C &= \tfrac{1}{2} \\ A + C &= 0 \end{aligned}$$

adding $\rightarrow$ $2A = \frac{1}{2} \Rightarrow A = \frac{1}{4}$

~~subtracting $\rightarrow$ $-2C = \frac{1}{2} \Rightarrow C = -\frac{1}{4}$.~~

Thus:

$$\frac{1}{(t+1)^2(t-1)^2} = \frac{1/4}{t+1} + \frac{1/4}{(t+1)^2} + \frac{-1/4}{t-1} + \frac{1/4}{(t-1)^2}$$

As before, we can use this decomposition to evaluate the corresponding integral:

$$\int \frac{dt}{(t^2-1)^2} = \frac{1}{4}\left\{ \int \frac{dt}{t+1} + \underbrace{\int \frac{dt}{(t+1)^2}} - \int \frac{dt}{t-1} + \int \frac{dt}{(t-1)^2} \right\}.$$

$$\int \frac{du}{u^2} = \int u^{-2}\,du = \frac{u^{-1}}{-1} = -\frac{1}{u}$$

Thus

$$\int \frac{dt}{(t^2-1)^2} = \frac{1}{4}\left\{ \ln|t+1| - \frac{1}{t+1} - \ln|t-1| - \frac{1}{t-1} \right\} + C$$

$$= \frac{1}{4}\left\{ \ln\left|\frac{t+1}{t-1}\right| - \frac{(t-1)+(t+1)}{t^2-1} \right\} + C$$

$$= \frac{1}{4}\left\{ \ln\left|\frac{t+1}{t-1}\right| - \frac{2t}{t^2-1} \right\} + C.$$

Sometimes, a decomposition of a polynomial involves an irreducible quadratic factor. Here's an example:

EX: Evaluate $\int \frac{x^2-x+6}{x^3+3x}\,dx$

SOL

NOTE.

$$\frac{x^2-x+6}{x^3+3x} = \frac{x^2-x+6}{x\underbrace{(x^2+3)}_{\text{irreducible quadratic function}}} = \frac{A}{x} + \frac{\boxed{Bx+C}}{x^2+3}$$

$\Rightarrow \quad x^2 - x + 6 = A(x^2+3) + (Bx+C)x$

$x=0 \Rightarrow \quad 6 = 3A \quad \Rightarrow A = 2$

coeff of x^2: $\quad 1 = A + B \quad \Rightarrow B = -1$

coeff of x^1: $\quad -1 = C \quad \Rightarrow C = -1.$

Thus:

$$\int \frac{x^2 - x + 6}{x^3 + 3x}\,dx = \underbrace{\int \frac{2}{x}\,dx}_{2\ln|x|} + \underbrace{\int \frac{-x-1}{x^2+3}\,dx}_{-\int \frac{x\,dx}{x^2+3} - \int \frac{dx}{x^2+3}}$$

$$-\underbrace{\int \frac{x\,dx}{x^2+3}}$$

$u = x^2+3$

$du = 2x\,dx$

$$-\int \frac{\frac{1}{2}du}{u} = -\frac{1}{2}\ln|x^2+3|$$

$$-\underbrace{\int \frac{dx}{x^2+3}} = -\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{x}{\sqrt{3}}\right)$$

(EQ 10 p497 of TEXT.)