L12
§7.3

Trigonometric Substitution

Advice: Understand evaluation of $\int\sqrt{a^2-x^2}\,dx$ thoroughly.

Consider

$$\int x\sqrt{a^2-x^2}\,dx. \qquad (*)$$

By now, your knee-jerk reaction would be to use the substitution rule:

$$u = a^2-x^2$$

$$du = -2x\,dx.$$

$$\int x\sqrt{a^2-x^2}\,dx = -\tfrac{1}{2}\int\sqrt{u}\,du = -\tfrac{1}{2}\frac{u^{3/2}}{3/2}+c.$$

$$= -\tfrac{1}{3}(a^2-x^2)^{3/2}+c.$$

But what if I removed "x" from the integrand in $(*)$:

$$\int\sqrt{a^2-x^2}\,dx$$

The substitution $u=a^2-x^2$ no longer works, mainly because it is not easy to write dx in terms of du.

Trigonometric substitution gets around this difficulty by doing substitution "the other way round". Instead of writing the sub. var. as a function of x, we write x as a function of the sub. variable. That variable is always an angle and the function is always a trigonometric function.

In this case, we set

$$x = a\sin\theta.$$

$$\Rightarrow \sqrt{a^2-x^2} = \sqrt{a^2-a^2\sin^2\theta} = \sqrt{a^2(1-\sin^2\theta)}$$

$$= \sqrt{a^2\cos^2\theta} = a\cos\theta.$$ [We will later restrict θ s.t. $\cos\theta \geq 0$].

Then, to complete the conversion of the integral to a new variable, we compute dx in terms of θ and $d\theta$, which is easy to do since we wrote x as a function of θ to begin with:

$$x = a\sin\theta$$

$$\Rightarrow dx = a\cos\theta\, d\theta$$

Thus:

$$\int \sqrt{a^2-x^2}\,dx = \int a\cos\theta \cdot a\cos\theta\, d\theta$$

$$= a^2\int \cos^2\theta\, d\theta$$

This is a trigonometric integral, which we may compute using the half-angle identity:

$$\cos^2\theta = \tfrac{1}{2}[1+\cos(2\theta)]$$

Thus:

$$\int \sqrt{a^2 - x^2}\, dx = a^2 \tfrac{1}{2} \int [1 + \cos(\underbrace{2\theta}_{u})]\, d\theta$$

$$= a^2 \tfrac{1}{2} \cdot \tfrac{1}{2} \int [1 + \cos u]\, du$$

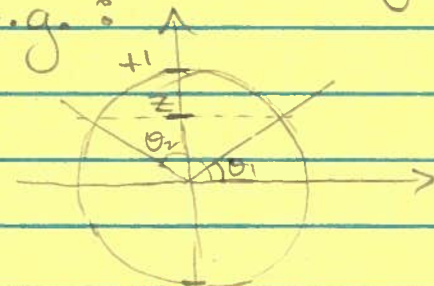
$$= \tfrac{1}{4} a^2 [u + \sin u] + c$$

$$= \tfrac{1}{4} a^2 [2\theta + \sin(2\theta)] + c \qquad (*)$$

At this point we would like to write θ in terms of x.

Recall: $x = a \sin\theta \Rightarrow \sin\theta = \frac{x}{a} \Rightarrow \theta = \sin^{-1}\left(\frac{x}{a}\right)$ (1)

Wait! There has been a sleight of hand here! We need to <u>define</u> $\sin^{-1}(z)$ because there are many θ's whose sin is z, e.g.:



We restrict the allowed values of θ s.t. $\sin^{-1}(z)$ is unique. One common choice is $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$.

By the way, notice that $z = x/a$ must lie in the range $[-1, 1]$. You might wonder whether this is unnecessarily restrictive. It is not:

$$\left|\frac{x}{a}\right| < 1$$

$$\Rightarrow \quad |x| < |a|$$

which is exactly what we need to ensure that $\sqrt{a^2 - x^2}$ is real (not complex).

The other thing in (b) that needs to be written in terms of x is $\sin(2\theta)$. Here, we can make use of the trig identity:

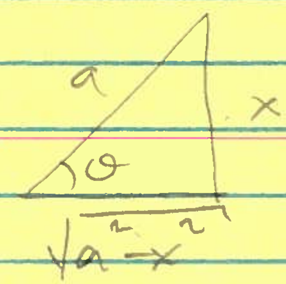
$$\sin(A+B) = \sin A \cos B + \cos A \sin B$$

$$\Rightarrow \quad \sin(2\theta) = 2\sin\theta\cos\theta$$

We now have a new problem: what is $\cos\theta$ in terms of x? Pictures are useful here. Recall:

$$x = a\sin\theta$$

$$\Rightarrow \sin\theta = \frac{x}{a} \Rightarrow$$



$$\Rightarrow \cos\theta = \frac{\sqrt{a^2-x^2}}{a}.$$

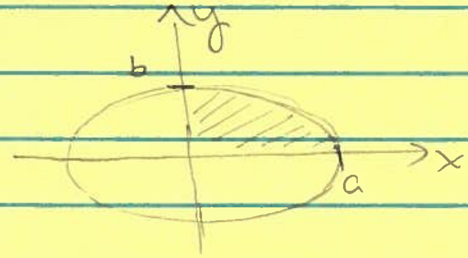
$$\sin(2\theta) = 2\sin\theta \cdot \cos\theta = 2\frac{x}{a}\cdot\frac{\sqrt{a^2-x^2}}{a}. \qquad (2)$$

Inserting (1) and (2) into (b) we get:

$$\int\sqrt{a^2-x^2}\,dx = \tfrac{1}{2}a^2\sin^{-1}\left(\frac{x}{a}\right)$$
$$+\tfrac{1}{2}x\sqrt{a^2-x^2} + C.$$

$$\boxed{\text{Rule \#1} \quad \int\sqrt{a^2-x^2}\,dx : \text{ use } x = a\sin\theta \text{ (restrict } \theta) \text{ and } 1-\sin^2\theta = \cos^2\theta}$$

$\int\sqrt{a^2-x^2}\,dx$ arises when computing area of an ellipse:



$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

Solve for y: $\frac{y^2}{b^2} = 1-\frac{x^2}{a^2}$

$$= \frac{a^2-x^2}{a^2}.$$

$$\Rightarrow y = \pm\frac{b}{a}\sqrt{a^2-x^2}$$

$$\text{Area}(\bigcirc) = 4\,\text{Area}(\,\cdot\,) = 4\int_0^a dx\cdot\frac{b}{a}\sqrt{a^2-x^2}$$

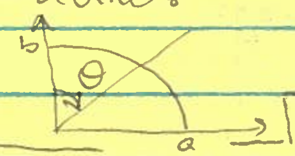
$$= 4\frac{b}{a}\int_0^a \sqrt{a^2-x^2}\,dx.$$

But we've shown that

$$\int \sqrt{a^2-x^2}\,dx = \frac{1}{4}a^2\left[2\theta+\sin(2\theta)\right] \quad [\text{cf } (*)]$$

where $x = a\sin\theta$.

Geometric Interpretation of $x = a\sin\theta$:



Thus:

$$\text{area(ellipse)} = 4\frac{b}{a}\cdot\frac{1}{4}a^2\left[2\theta+\sin(2\theta)\right]_{\theta(x=0)=0}^{\theta(x=a)=\pi/2}$$

$$= ab\left[(\pi+\sin(\pi)) - (0+\sin(0))\right]$$

$$= \pi ab. \qquad (**)$$

In particular, if $a=b=r$, then ellipse becomes a circle of radius r, and $(**)$ reduces to πr^2, as it must!

Recall that $x = a\sin\theta$ worked for $\sqrt{a^2-x^2}$ because $a^2 - a^2\sin^2\theta = a^2(1-\sin^2\theta) = a^2\cos^2\theta$.

There is a similar strategy for $\sqrt{a^2+x^2}$:

Rule 2 $\int \sqrt{a^2+x^2}\,dx$: use $x = a\tan\theta$
and $1+\tan^2\theta = \sec^2\theta$

Here's how it works:

$$a^2+x^2 = a^2(1+\tan^2\theta) = a^2\sec^2\theta.$$

$$dx = a\sec^2\theta\, d\theta.$$

$$\int \sqrt{a^2+x^2}\,dx = \int a\sec\theta \cdot a\sec^2\theta\, d\theta$$

$$= a^2 \underbrace{\int \sec^3\theta\, d\theta}_{\text{integrate by parts.}}$$

Recall from L11:

$$d(\tan\theta) = \sec^2\theta\, d\theta$$

$$d(\sec\theta) = \sec\theta \cdot \tan\theta\, d\theta.$$

$u = \sec\theta$ $\quad$ $dv = \sec^2\theta\, d\theta$

$du = \sec\theta\tan\theta\, d\theta$ $\quad$ $v = \tan\theta.$

Thus:

$$\int \sec^3\theta\, d\theta = \sec\theta\cdot\tan\theta - \int \tan^2\theta \cdot \sec\theta\, d\theta \quad (3)$$

To evaluate the 2^{nd} integral, remember that:

$$\tan^2\theta = \sec^2\theta - 1$$

so:

$$\int \tan^2\theta \sec\theta \, d\theta = \underbrace{\int \sec^3\theta \, d\theta}_{I} - \int \sec\theta \, d\theta \quad (4)$$

(3), (4) together give:

$$I = \sec\theta \tan\theta - I + \int \sec\theta \, d\theta$$

$$\Rightarrow \quad I = \tfrac{1}{2}\sec\theta\tan\theta + \tfrac{1}{2}\underbrace{\int \sec\theta \, d\theta}_{\ln|\sec\theta + \tan\theta| + c}$$

(TEXT p483; cf p10 of these notes)

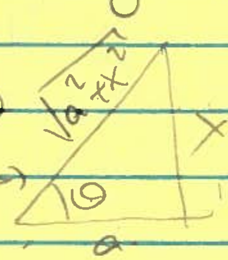
Thus:

$$\int \sqrt{a^2 + x^2}\, dx = \tfrac{1}{2}a^2 \{ \sec\theta \tan\theta + \ln|\sec\theta + \tan\theta| + c \}$$

Finally lets write RHS in terms of x:

$x = a\tan\theta$

$\Rightarrow \tan\theta = \frac{x}{a} \Rightarrow$



$$\Rightarrow \sec\theta \tan\theta = \frac{\sin\theta}{\cos^2\theta}$$

$$= \frac{x}{\sqrt{a^2+x^2}} \cdot \frac{a^2+x^2}{a^2}$$

$$= \frac{x}{a^2}\sqrt{a^2+x^2}$$

$$\sec\theta + \tan\theta = \frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a} = \frac{\sqrt{a^2+x^2}+x}{a}$$

$$\Rightarrow \ln|\sec\theta + \tan\theta| = \ln\left|\sqrt{a^2+x^2}+x\right| - \ln a$$

Thus:

$$\int \sqrt{a^2+x^2}\,dx = \frac{1}{2}x\sqrt{a^2+x^2} + \frac{1}{2}a^2 \ln\left|\sqrt{a^2+x^2}+x\right| + C,$$

"absorbs $-\ln a$".

Your homework: know how to deal with integrals containing $\sqrt{x^2-a^2}$ → see Example 5 §7.3 p 489 of TEXT.

Pf of: $\int \sec\theta\, d\theta = \ln|\sec\theta + \tan\theta| + C$.

$$\int \sec x\, dx = \int \sec x \cdot \frac{\sec x + \tan x}{\sec x + \tan x}\, dx$$

$$= \int \frac{\sec^2 x + \sec x \tan x}{\sec x + \tan x}\, dx$$

$u = \sec x + \tan x$

$du = (\sec x \tan x + \sec^2 x)\, dx$

$$\int \sec x\, dx = \int \frac{du}{u}$$

$$= \ln|u| + C$$

$$= \ln|\sec x + \tan x| + C.$$