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L11
§7.2

Trigonometric Integrals

Ex Evaluate $\int \cos^3 x\,dx$.

SoL $u = \cos x \Rightarrow du = -\sin x\,dx$ ... but there is no $\sin x$ factor!

$u = \sin x \Rightarrow du = \cos x\,dx$?
Well, there is a $\cos x$ factor:

$$\int \cos^2 x \cdot \underbrace{\cos x\,dx}_{du}$$

What should we do next? Write $\cos^2 x$ in terms of $\sin x$, using the trig id:

$$\cos^2 x + \sin^2 x = 1.$$

ie.

$$\int \underbrace{(1-\sin^2 x)}_{1-u^2} \cdot \underbrace{\cos x\,dx}_{du}$$

$$= \int (1-u^2)\,du = u - \frac{u^3}{3} + C = \sin x - \frac{1}{3}\sin^3 x + C$$

Strategy 1 Write integrand as product of $\cos x$ and another factor written in terms of $\sin x$, or vice versa

Here's another example of that strategy:

Ex Evaluate $\int \sin^5 x \cos^2 x \, dx$

SOL Try: $\int \underbrace{\sin^5 x \cos x}_{\downarrow} \cdot \underbrace{\cos x \, dx}_{du}$

Want to write in terms of $u = \sin x$.

Recall: $\sin A \cos B = \frac{1}{2}\left[\sin(A-B) + \sin(A+B)\right]$

BTW This identity useful to evaluate $\int \sin(\alpha x)\cos(\beta x)\,dx$, cf. Ex 9 p 484

$\Rightarrow$ $\sin^5 x \cos x = \sin^4 x \cdot \sin x \cos x$

$$= \sin^4 x \cdot \frac{1}{2}\left[\sin(0) + \sin(2x)\right]$$

$$= \frac{1}{2}\sin^4 x \sin(2x)$$

Close, but not good enough!

Try: $\int \underbrace{\sin^4 x \cos^2 x}_{\downarrow} \underbrace{\sin x \, dx}_{du}$

Want to write this in terms of $u = -\cos x$.

$\sin^4 x \cos^2 x = (\sin^2 x)^2 \cos^2 x$

$$= (1 - \cos^2 x)^2 \cos^2 x$$

$$= (1-u^2)^2 u^2$$

$$= (1 + u^4 - 2u^2)u^2 = u^2 + u^6 - 2u^4$$

$\therefore \int \sin^4 x \cos^2 x \cdot \sin x\, dx$

$= \int (u^2 + u^6 - 2u^4)\, du$

$= \frac{1}{3}u^3 + \frac{1}{7}u^7 - 2 \cdot \frac{1}{5} u^5 + C$

$= -\frac{1}{3}\cos^3 x - \frac{1}{7}\cos^7 x + \frac{2}{5}\cos^5 x + C.$

Strategy 1 worked for the previous examples because there was an odd power of $\cos x$ or $\sin x$. So how do we tackle the following example:

EX Evaluate $\int_0^{\pi} \sin^2 x\, dx$

SOL $\int_0^{\pi} \underbrace{\sin x}_{\downarrow} \cdot \underbrace{\sin x\, dx}_{du}$

cannot write in terms of $u = -\cos x$.

Recall: $\sin A \sin B = \frac{1}{2}[\cos(A-B) - \cos(A+B)]$

$\Rightarrow \sin^2 x = \frac{1}{2}[\cos(0) - \cos(2x)]$

$= \frac{1}{2}[1 - \cos(2x)]$

This is just what we need!

$\int \sin^2 x\, dx = \frac{1}{2}\int [1 - \cos(2x)]\, dx$

$u = 2x$ $\quad \frac{1}{2} \cdot \frac{1}{2} \int [1 - \cos u] \, du$

$du = 2dx$

$$= \frac{1}{4} [u - \sin u]$$

$$= \frac{1}{4} [2x - \sin(2x)]_0^{\pi}$$

$$= \frac{1}{4} [(2\pi - \sin(2\pi)) - (0 - \sin(0))]$$

$$= \frac{\pi}{2}$$

STRATEGY 2 Use half-angle identities:

$$\sin^2 x = \frac{1}{2}[1 - \cos(2x)] \; ; \; \cos^2 x = \frac{1}{2}[1 + \cos(2x)].$$

Some trickier applications of these strategies:

EX Evaluate $\int \sqrt{\cos\theta} \sin^3\theta \, d\theta$.

SOL

$$\left.\begin{array}{l} \int \underbrace{\sqrt{\cos\theta} \sin^2\theta}_{\downarrow} \cdot \underbrace{\sin\theta \, d\theta}_{-du}. \\ \text{write in terms of } u = \cos\theta \end{array}\right\} \text{Strategy 1}$$

$$\sqrt{\cos\theta} \sin^2\theta = \sqrt{\cos\theta} (1 - \cos^2\theta) = u^{1/2}(1 - u^2)$$

Thus integral is:

$$-\int (u^{1/2} - u^{5/2})\, du$$

$$= -\frac{u^{3/2}}{3/2} + \frac{u^{7/2}}{7/2} + C$$

$$= -\frac{2}{3}\cos^{3/2}\theta + \frac{2}{7}\cos^{7/2}\theta + C.$$

Ex Evaluate $\displaystyle\int \frac{\sin^2(1/t)}{t^2}\, dt.$

Sol $x = 1/t.$ Integral is:

$dx = -\dfrac{dt}{t^2}$ $\qquad$ $\underbrace{-\int \sin^2 x\, dx. \quad \text{cf. Ex on p. 3}}_{\text{strategy 2}}$

Ex $\int t \sin^2 t\, dt$

This is similar in appearance to $\int \sin^2 t\, dt$, so lets tackle it the same way (Strategy 2).

$$\int t\sin^2 t\, dt = \frac{1}{2}\int t[1-\cos(2t)]\, dt.$$

$$= \frac{1}{2}\frac{t^2}{2} - \frac{1}{2}\underbrace{\int t\cos(2t)\, dt}_{\text{integrate by parts (cf L10)}}.$$

$u = t$; $dv = \cos(2t)\, dt$ $\qquad$ $\int t\cos(2t)\, dt$

$du = dt$; $v = \frac{1}{2}\sin(2t)$ $\qquad$ $= t\cdot\frac{1}{2}\sin(2t) - \int \frac{1}{2}\sin(2t)\cdot dt.$

$$= \frac{1}{2}t \cdot \sin(2t) + \frac{1}{2} \cdot \frac{1}{2}\cos(2t) + C.$$

Thus

$$\int t\sin^2 t\,dt = \frac{1}{4}t^2 - \frac{1}{4}t\sin(2t) - \frac{1}{8}\cos(2t) + C.$$

We were able to tackle integrals of the form $\int \cos^m x \sin^n x\,dx$ because of

$$\left.\begin{aligned} d(\cos x) &= -\sin x\,dx \\ d(\sin x) &= \cos x\,dx \\ \cos^2 x + \sin^2 x &= 1 \end{aligned}\right\} \text{Strategy 1.}$$

There is a similar strategy for integrals of the form $\int \tan^m x \sec^n x\,dx$ because:

$$d(\tan x) = \sec^2 x\,dx$$

$$d(\sec x) = \sec x \tan x\,dx.$$

$$\tan^2 x - \sec^2 x = -1. \quad (\text{or } \sec^2 x = 1 + \tan^2 x).$$

EX Evaluate $\int \tan x \sec^3 x \, dx$.

SOL Try: $\int \underbrace{\tan x \sec x}_{\downarrow} \cdot \underbrace{\sec^2 x \, dx}_{du}$

don't know how to write in terms of $u = \tan x$.

Try: $\int \underbrace{\sec^2 x}_{u^2} \cdot \underbrace{\sec x \tan x \, dx}_{du}$

$= u^3/3 + C$

$= \frac{1}{3}\sec^3 x + C.$

EX Evaluate $\int \tan^2 x \sec x \, dx$

SOL Try: $\int \underset{\downarrow}{\tan x} \cdot \underbrace{\tan x \sec x \, dx}_{du}$

? in terms of $u = \sec x$.

Try: $\int \underbrace{\frac{\tan^2 x}{\sec x}} \underbrace{\sec^2 x \, dx}_{du}$

? in terms of $u = \tan x$.

We need a new strategy.

Convert everything to secx:

$$\int \tan^2 x \cdot \sec x \, dx$$

$$= \int (\sec^2 x - 1) \sec x \, dx$$

$$= \underbrace{\int \sec^3 x \, dx}_{\text{integrate by parts}} - \int \sec x \, dx. \qquad (\Delta)$$

$u = \sec x$; $dv = \sec^2 dx$

$du = \sec x \tan x \, dx$; $v = \tan x$

$$\therefore \int \sec^3 x \, dx = \sec x \cdot \tan x - \underbrace{\int \tan^2 x \cdot \sec x \, dx}_{I}$$

Sub into (Δ) to get:

$$I = \sec x \tan x - I - \int \sec x \, dx$$

$$\Rightarrow \quad 2I = \sec x \tan x - \int \sec x \, dx.$$

Turns out that:

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C \qquad (\text{cf } p483)$$

Thus:

$$I = \frac{1}{2} \sec x \tan x - \frac{1}{2} \ln|\sec x + \tan x| + C.$$