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§7.1

Integration by Parts.

Note: please try to know as many of the integrals on p471 as possible (I know it's a pain)!

Every differentiation rule has a corresponding integration rule. The Substitution Rule corresponds to the Chain Rule. We shall see in this lecture that the product rule of differentiation leads to a rule in integration called integration by parts.

Recall the product rule:

$$(fg)' = f'g + fg'$$

This says that the anti-derivative of $f'g+fg'$ is fg. Therefore

$$\int (f'g+fg')\,dx = fg + c.$$

$$\Rightarrow \int fg'\,dx = fg - \int f'g\,dx + c.$$

We can get rid of the "c" since a constant would emerge anyway when evaluating $\int f'g\,dx$. Thus:

$$\int fg'\,dx = fg - \int f'g\,dx$$

This formula becomes easier to remember if we make a small substitution:

$$u = f \; ; \; v = g$$

Then

$$du = f'dx \; ; \; dv = g'dx.$$

In terms of u and v, (*) is:

$$\boxed{\int u\,dv = uv - \int v\,du}$$ Integration by parts.

EX Evaluate

$$\int x e^x dx.$$

SOL

$$u = x \qquad dv = e^x dx$$

$$du = dx \qquad v = \int dv = \int e^x dx = e^x$$

(we may choose any antiderivative of v')

$$\therefore \int x e^x dx = \underbrace{x}_{u} \cdot \underbrace{e^x}_{v} - \int \underbrace{e^x}_{v} \underbrace{dx}_{du}$$

$$= x e^x - e^x + C.$$

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What dictates our choice of u and v?
For example, in the previous example, why didn't we choose

$$u = e^x \quad ; \quad dv = x\,dx \qquad ?$$

Well, let's work it out!

$$du = e^x\,dx \quad ; \quad v = \int x\,dx = \tfrac{1}{2}x^2$$

$$\int x e^x\,dx = \underbrace{e^x}_{u} \cdot \underbrace{\tfrac{1}{2}x^2}_{v} - \int \underbrace{\tfrac{1}{2}x^2}_{v} \cdot \underbrace{e^x\,dx}_{du}.$$

$$= \tfrac{1}{2}x^2 e^x - \tfrac{1}{2}\int x^2 e^x\,dx.$$

We have gotten an even more difficult integral to evaluate! ($\int x^2 e^x\,dx$).

The choice of u and v is dictated by our goal of obtaining a simpler integral than the one we started out with.

Ex Evaluate

$$\int \ln x\,dx$$

SOL: $\quad u = \ln x \qquad dv = dx$

$du = \frac{1}{x}dx \qquad v = x$

$$\int \ln x\,dx = \underbrace{\ln x}_{u} \cdot \underbrace{x}_{v} - \int \underbrace{x}_{v} \cdot \underbrace{\tfrac{1}{x}dx}_{du}$$
$$= x\ln x - x + C.$$

Sometimes we need to integrate by parts twice:

Ex Evaluate
$$\int x^2 e^x dx$$

Sol
$$u = x^2 \;;\; dv = e^x dx \qquad (1)$$
$$du = 2x\,dx \;;\; v = e^x. \qquad (2)$$

Notice that (2):
$$v\,du = 2xe^x dx$$
is simpler than (1):
$$u\,dv = x^2 e^x dx.$$

Thus:
$$\int x^2 e^x dx = x^2 \cdot e^x - \int e^x \cdot 2x\,dx$$
$$= x^2 e^x - 2\int x e^x dx \qquad (3)$$

Though simpler, $\int x e^x dx$ is still not obvious. Let's tackle it by the same method.

$$u = x \qquad\qquad dv = e^x dx$$
$$du = dx \qquad\qquad v = e^x.$$

$$\int x e^x dx = xe^x - \int e^x \cdot dx$$
$$= xe^x - e^x + C.$$

Substitute into (3) to get:

$$\int x^2 e^x dx = x^2 e^x - 2xe^x + 2e^x + D.$$

Ex Evaluate

$$\int e^x \sin x \, dx$$

SoL

$$u = e^x \qquad dv = \sin x \, dx$$
$$du = e^x dx \qquad v = -\cos x$$

$$\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx. \tag{4}$$

$$u = e^x \qquad dv = \cos x \, dx$$
$$du = e^x dx \qquad v = \sin x$$

$$\int e^x \cos x = e^x \sin x - \int e^x \sin x \, dx \tag{5}$$

Sub (5) in (4):

$$\int e^x \sin x\, dx = -e^x \cos x + e^x \sin x - \int e^x \sin x\, dx$$

This is an equation for $\int e^x \sin x\, dx$, which can be solved as follows:

$$2\int e^x \sin x\, dx = e^x(\sin x - \cos x)$$

$$\int e^x \sin x\, dx = \tfrac{1}{2} e^x (\sin x - \cos x) + c$$

Integration by parts works for definite integrals too!

<u>Ex</u> Compute

$$\int_0^1 \arctan x\, dx$$

<u>Sol</u>

$u = \arctan x$ $\qquad$ $dv = dx$

$du = \dfrac{1}{1+x^2}\, dx$ $\qquad$ $v = x.$

$$\int \arctan x\, dx = x \arctan x - \int \frac{x\,dx}{1+x^2} \qquad (6)$$

$$\left.\begin{array}{l} u = 1+x^2 \\ du = 2x\,dx \end{array}\right\} \Rightarrow \int \frac{x\,dx}{1+x^2} = \int \frac{\frac{1}{2}du}{u} = \frac{1}{2}\ln u =$$

$$= \frac{1}{2}\ln(1+x^2) \qquad (7)$$

Sub (7) in (6):

$$\int_0^1 \arctan x\,dx = \Big[x \cdot \arctan x\Big]_0^1 - \Big[\tfrac{1}{2}\ln(1+x^2)\Big]_0^1$$

But

$\tan 0 = 0 \Rightarrow \arctan 0 = 0.$

$\tan(\pi/4) = 1 \Rightarrow \arctan 1 = \pi/4$

$\ln(1+1^2) = \ln 2$

$\ln(1+0^2) = \ln 1 = 0.$

Thus:

$$\int_0^1 \arctan x \cdot dx = \frac{\pi}{4} - \frac{1}{2}\ln 2$$

Don't forget the substitution rule, which may be combined w/ integration by parts.

EX Evaluate

$$\int e^{\sqrt{x}}\,dx$$

Sol $t = \sqrt{x} \Rightarrow dt = \frac{1}{2}x^{-1/2}\,dx.$

$\Rightarrow 2\sqrt{x}\,dt = dx$

ie. $2t\,dt = dx.$

$$\therefore \int e^{\sqrt{x}}\,dx = \int e^t \cdot 2t\,dt = 2\underbrace{\int te^t\,dt}_{\text{we computed this on p2!}}.$$

Ex Evaluate:

$$\int x \ln(1+x)\,dx$$

Sol $y = 1+x \qquad dy = dx.$

$$\int x \ln(1+x)\,dx = \int (y-1)\ln y\,dy$$

$$u = \ln y \qquad dv = (y-1)\,dy$$

$$du = \frac{1}{y}\,dy \qquad v = \frac{1}{2}y^2 - y$$

$$\int (y-1)\ln y\,dy = \left(\tfrac{1}{2}y^2 - y\right)\ln y - \int \left(\tfrac{1}{2}y^2 - y\right)\cdot \frac{1}{y}\,dy$$

$$= y\left(\tfrac{1}{2}y - 1\right)\ln y - \underbrace{\int \left(\tfrac{1}{2}y - 1\right) dy}_{\frac{1}{2}\frac{y^2}{2} + y + C}$$

Thus:

$$\int x \ln(1+x)\,dx = (1+x)\left[\tfrac{1}{2}(1+x) - 1\right]\ln(1+x)$$

$$- \tfrac{1}{4}(1+x)^2 + (1+x) + C$$

which can be simplified somewhat.