2B

- www.math.uci.edu/undergraduate/courses/calculus-2a2b-resources
- class website posted soon.

L1 §4.9

Antiderivatives (Review)

Definition A function F is called an antiderivative of f on an interval I if $F'(x) = f(x)$ for all x on I.

Example Let $f(x) = x^2$. What is it's antiderivative?

Sol^n We want an F s.t. $F'(x) = x^2$.

Recall that if $F(x) = \alpha x^n$ then $F'(x) = \alpha r x^{r-1}$ (Power Rule).

What must α, r be to satisfy

$$\alpha r x^{r-1} = x^2 ?$$

Answer:

$$r = 3$$
$$\alpha = 1/3.$$

ie. $F(x) = \frac{1}{3}x^3$

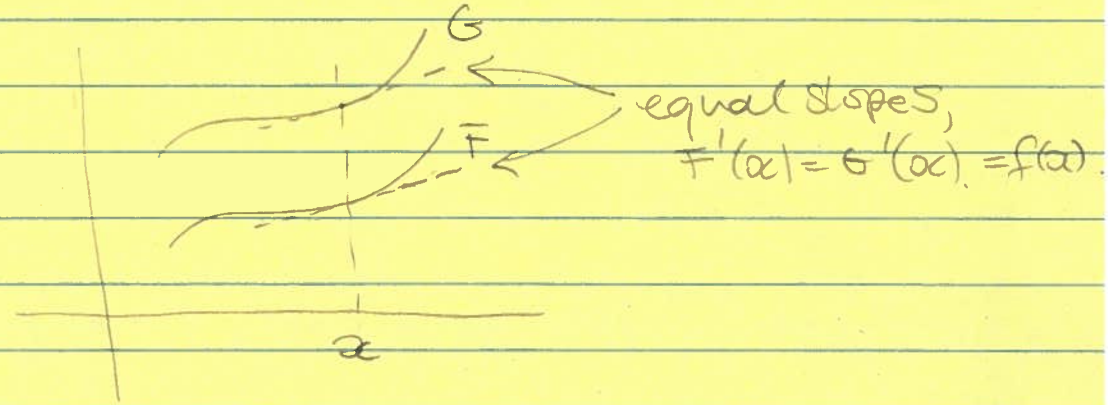
But $G(x) = \frac{1}{3}x^3 + 100$ also works because

$$\frac{d}{dx} 100 = 0.$$

Thus: both F and G are antiderivati[ves] of f; and $G = F +$ constant.

In fact, this is general: once you've found one antiderivative, you've found them all ... by adding various constants.

We can draw a picture to illustrate this:



Antiderivatives just "reverse" differentiation, eg.

$$\ln x \xrightarrow{d/dx} 1/x \qquad (1/x \to \ln x: \text{antiderivative})$$

$$e^x \xrightarrow{d/dx} e^x \qquad (e^x \to e^x: \text{antiderivative})$$

$$b^x \xrightarrow{d/dx} \ln b \cdot b^x$$
$$b^x \xleftarrow{\text{antiderivative}} \ln b \cdot b^x$$

ASIDE: Recall, to compute $d/dx\,(b^x)$, let

$$y = b^x$$

$$\Rightarrow \ln y = x \ln b$$

$$\Rightarrow \frac{d}{dy} \ln y = \frac{d}{dx}(x \ln b) \quad \text{[implicit differentiat}^{\text{n}}\text{]}$$

$$\Rightarrow \frac{1}{y}\frac{dy}{dx} = \ln b$$

↑ chain rule

$$\Rightarrow \frac{dy}{dx} = y \ln b$$

$$\text{or} \quad \frac{d}{dx}(b^x) = b^x \ln b.$$

<u>EX</u> Find $F(x)$ s.t. $F'(x) = 5x^4 - 2x^5$ and $F(0) = 4$.

<u>Sol$^{\text{n}}$</u>

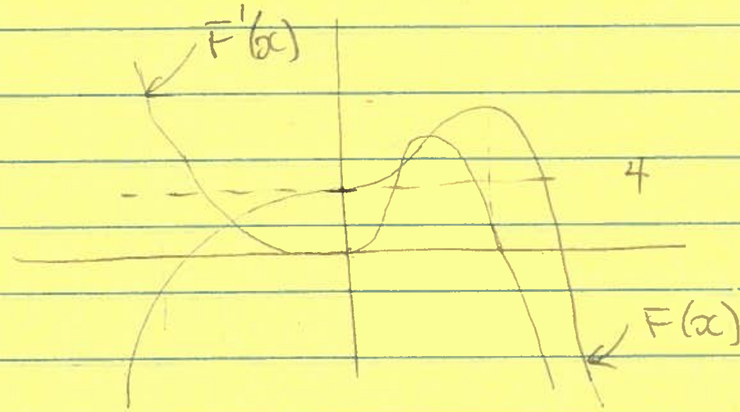
$$F(x) = 5 \cdot \frac{x^5}{5} - 2\frac{x^6}{6} + C$$

$$= x^5 - \frac{1}{3}x^6 + C$$

$$F(0) = 4 \Rightarrow 0^5 - \frac{1}{3}0^6 + C = 4 \Rightarrow C = 4$$

So $F(x) = x^5 - \frac{1}{3}x^6 + 4$

Pictorially:



Note that: $F'(x) = 0$ where $F(x)$ has a local maximum; $F'(x) > 0$ where $F(x)$ is increasing, and vice versa.

What we did in the previous example is to solve a differential equation: an equation involving the derivative of a function.

A differential equation may involve more than one derivative:

Ex Find f if $f''(\theta) = \sin\theta + \cos\theta$, $f(0) = 3$, $f'(0) = 4$.

Sol^n The antiderivative of $f''(\theta)$ is:

$$f'(\theta) = -\cos\theta + \sin\theta + C$$

$f'(0) = 4 \Rightarrow -\cos(0) + \sin(0) + C = 4$

$\Rightarrow -1 + 0 + C = 4$

$\Rightarrow C = 4 + 1 = 5.$

$\Rightarrow f'(\theta) = -\cos\theta + \sin\theta + 5$

The antiderivative of $f'(\theta)$ is

$f(\theta) = -\sin\theta - \cos\theta + 5\theta + D$

But $f(0) = 3 \Rightarrow$

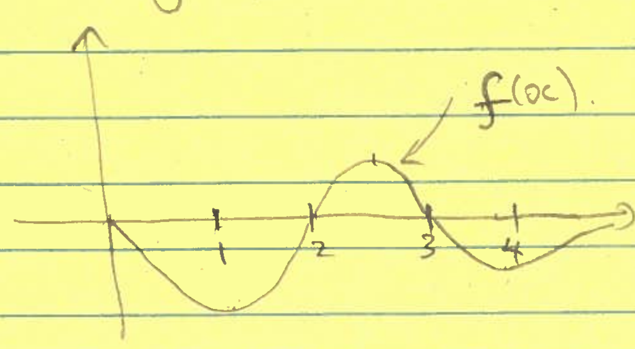
$-\sin(0) - \cos(0) + 5(0) + D = 3$

$\Rightarrow 0 - 1 + 0 + D = 3$

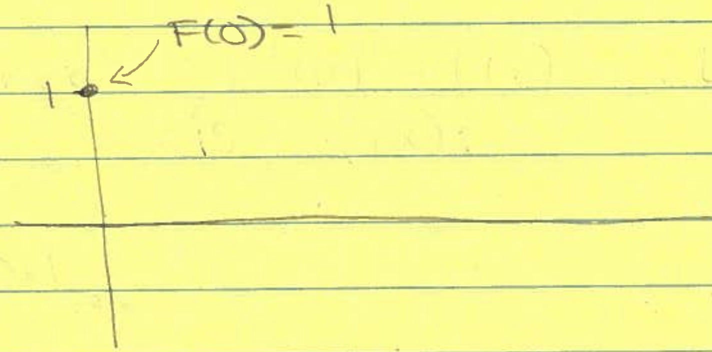
$\Rightarrow D = 3 + 1 = 4.$

$\Rightarrow f(\theta) = -\sin\theta - \cos\theta + 5\theta + 4.$

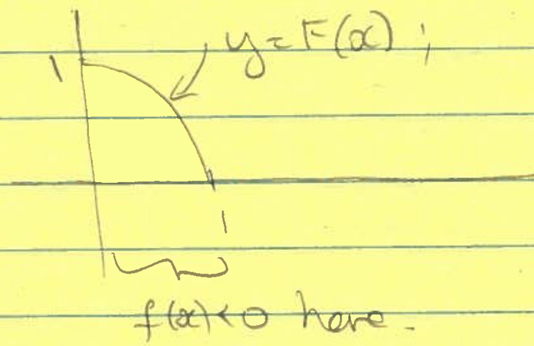
EX: Sketch $F(x)$ if $F'(x) = f(x)$, $F(0) = 1$, and the sketch of $f(x)$ is:



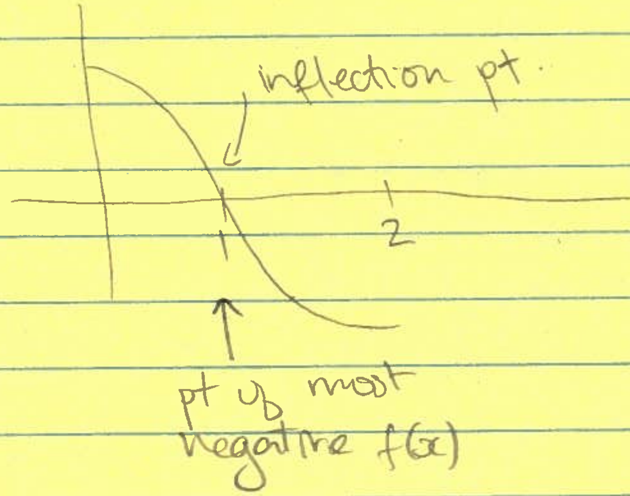
Sol^n



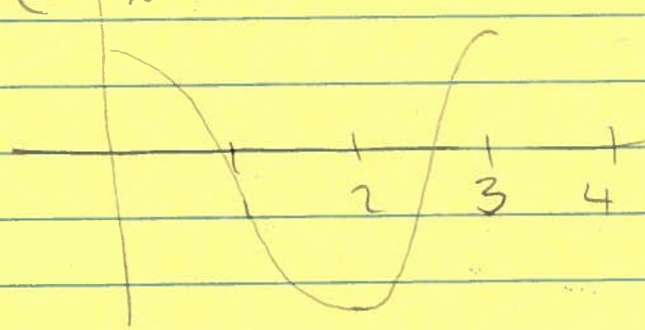
Since $F'(x)=f(x)$, the slope of $F(x)$ is just $f(x)$:



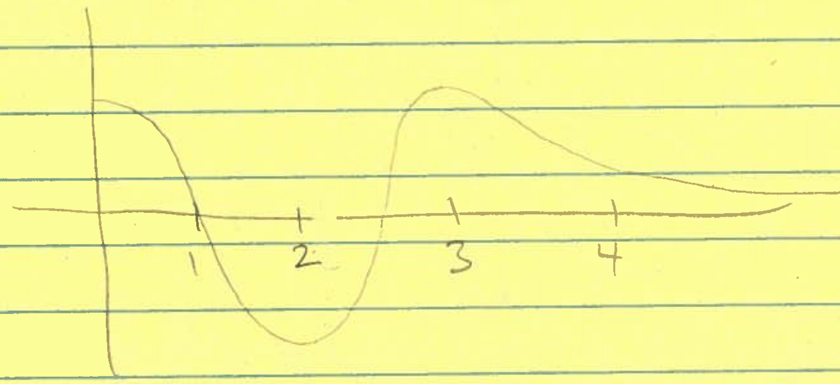
At $x=1$, $f(x)$ is at a minimum, ie the slope of F is at it's steepest, ie F is @ an inflection pt:



At $x=2$, $f(x)=0$, ie slope of $F(x)$ is zero $\Rightarrow$ $F(x)$ reaches local minimum, followed by inflection @ $x = 2.5$, and local max @ $x = 3$.



Finally: $F(x)$ declines for $x > 3$ ($f(x) < 0$) with inflection @ $x = 4$.



Here's an example where differential equations arise in physics:

object position $= s(t)$

" velocity $= v(t) = ds/dt$.

" acceleration $= a(t) = dv/dt$.

Ex $a(t) = 2t + 1$, $s(0) = 3$, $v(0) = -2$. Find $s(t)$.

Sol^{n} $a(t) = 2t+1$

$\Rightarrow \quad v' = 2t+1$

$\Rightarrow \quad v = 2\frac{t^2}{2} + t + c$

but $\quad v(0) = -2 \Rightarrow \quad c = -2$

$\Rightarrow \quad v = t^2 + t - 2$

$\Rightarrow \quad s' = t^2 + t - 2$

$\Rightarrow \quad s = \frac{t^3}{3} + \frac{t^2}{2} - 2t + D$

but $\quad s(0) = 3 \Rightarrow \quad D = 3$

$\Rightarrow \quad s = \frac{t^3}{3} + \frac{t^2}{2} - 2t + 3.$