Math 175: Combinatorics
Final Exam Practice Problems

1 True/False

1. For a fixed positive integer n define
$$F(t) = \binom{2n}{0} + \binom{2n}{1} + \cdots + \binom{2n}{t-1}.$$
Then for any n and all t satisfying $1 \le t \le n+1$,
$$F(t) \le \frac{t}{2} \cdot \binom{2n}{n}.$$

2. Let $F_0 = 0,\ F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \ge 2$. Then for all $n,\ F_n < \left(\frac{3}{2}\right)^n$.

3. $F_0 = 0,\ F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$ for $n \ge 2$. Let $C_n = \frac{1}{n+1}\binom{2n}{n}$ denote the n^{th} Catalan number. Then $C_n = o(F_{2n})$.

4. We have
$$\binom{n}{5} \sim \frac{n^5}{5!}.$$

2 Short Answer

1. For which positive integers n and k is $\binom{n}{k+1} = 2\binom{n}{k}$?

2. How many permutations of $\{1, 2, \ldots, 9\}$ have exactly three cycles of length 3?

3. A town has recently constructed ten new intersections. Some of these will get traffic lights, and some of those that get traffic lights will also get a gas station. In how many different ways can this happen?

4. How many six-digit positive numbers are there where the sum of the digits is at most 51?

5. In how many ways can the elements of $\{1, 2, \ldots, n\}$ be permuted so that 1 comes before 2 and 3?

3 Problems

1. Take the numbers $1, 2, \ldots, 10$ and put them around a circle (in some order). Prove that, no matter how you arrange them, there will be three consecutive numbers such that their sum is at least 17.
For example, if you place the numbers in increasing order, $\{9, 10, 1\}$ is such a consecutive triple.

2. Prove that for all integers $n \geq 2$,

$$2^{n-2} \cdot n \cdot (n-1) = \sum_{k=2}^{n} k(k-1)\binom{n}{k}.$$

3. At a tennis tournament there were 2^n participants and any two of them had played against each other exactly one time. Show that we can find $n+1$ players who can form a line so that everyone has defeated all of the players who are behind her in line.

4. Prove for all positive integers n,

$$n \cdot \binom{2n-1}{n-1} = \sum_{k=1}^{n} k \cdot \binom{n}{k}^2.$$

5. Show that every positive integer n possesses a representation

$$n = \sum_{k \geq 1} a_k k!,$$

with $0 \leq a_k \leq k$.

6. A store has n different products for sale. Each of them has a different cost that is at least one dollar, at most n dollars, and is a whole number. A customer is allowed to inspect exactly k different items. After doing so, he purchases the least expensive of those k items he inspected. Show that on average he will pay $\frac{n+1}{k+1}$ dollars.

   Note: This problem is hard. You might want to try to do this one last after you've done the others.

7. Assume that a positive integer cannot have 0 as its leading digit.
   (a) How many five-digit positive numbers have no repeated digits at all (for example, 12345 but not 12341)?
   (b) How many have no consecutive repeated digits (for example, 12341 but not 12331)?
   (c) How many have at least one run of consecutive repeated digits (for example, 11234, 22323, 45551, or 11551, but not 12121)?

8. (a) Find the smallest positive integer m such that $m^2 < 2^{m-1}$.
   (b) Let m be the number from part (a). Prove the following statement by induction. For every $n \geq m$, $n^2 < 2^{n-1}$.

9. Robin Hood shoots arrows at a target. The target is an equilateral triangle of side length 1. Robin Hood never misses the target. When an arrow hits the target, it stays there.
   (a) Robin Hood has shot 5 arrows. Prove that there are two arrows such that the distance between them does not exceed 1/2.

(b) Suppose that $n > 2$ and that Robin Hood has shot $n^2 + 1$ arrows. Prove that there are two arrows such that the distance between them does not exceed $1/n$.

10. Prove that for any positive integer $n \geq 2$,

$$\binom{2n}{n} \leq 3 \cdot 2^{2n-3}.$$

11. Determine which of the following statements are true:

(a) $n! \sim \left(\frac{n+1}{2}\right)^n$,

(b) $n! \sim ne(n/e)^n$,

(c) $n! = o((n/e)^n)$,

(d) $\ln(n!) \sim n \cdot \ln(n)$.

12. Fix a positive integer n and consider

$$\binom{n}{k+1} - \binom{n}{k}.$$

For which value of k is this difference largest?

Hint: This problem is pretty challenging. One idea is to consider the difference of consecutive differences. That is, for which values of k is

$$\left(\binom{n}{k+1} - \binom{n}{k}\right) - \left(\binom{n}{k} - \binom{n}{k-1}\right) \geq 0?$$

13. (a) State Stirling's asymptotic formula for the size of $n!$.

(b) Use Stirling's formula to give an asymptotic formula for $\binom{2n}{n}$.

(c) Use Stirling's formula to give an asymptotic formula for

$$\frac{\binom{2n}{n-t}}{\binom{2n}{n}}.$$

Your answer should be a function for m and t and should not involve any factorials.

Note: When you plug in $t = 0$ for your formula in (c), you should definitely get 1.