Math 175: Combinatorics
Exam 2 Practice Problems

Since Homework 5 was posted we continued our discussion of inclusion-exclusion. We gave a proof similar to the one given for Theorem 2.6 of HHM (but our proof was more detailed).

We then started to discuss derangements and the n students, n lunchboxes problem. Derangements are covered in pages 160-161 of HHM.

We spent some time discussing the set of all permutations of $\{1, 2, \ldots, n\}$, which we denoted by S_n. We saw a few ways of writing a permutation: two line notation, one line notation, and cycle notation. Section 2.7.1 of HHM is on 'Permutation Groups' has a nice explanation of these concepts. This section contains some additional material that we will not need in this course (for example, you do not need to know what a group is).

On Monday we will finish up the proof of our formula for $D(n)$, the number of derangements of $\{1, 2, \ldots, n\}$. We will then discuss the number of permutations with a given cycle structure. After that we will return to the problem of 100 prisoners who need to find their names in 100 boxes that we discussed in the first lecture.

1 True/False

1. At a party with 10 guests there are at least two people who know the same number of other guests.

   Note: If person A 'knows' person B, then person B must also 'know' person A.

2. Let C_k denote the kth Catalan number. For $k \geq 1$,

$$C_k = \frac{2^k}{(k+1)!} \prod_{i=1}^{k} (2i - 1).$$

3. A professor has been working for the same department for 30 years. There are two semesters in a year. She taught two courses in each semester. The department offers 15 different courses. There must have been two semesters when this professor taught exactly the same pair of courses.

2 Problems

1. A k-digit number cannot have 0 as its first digit (for example 09 is not a two-digit number.) A number is a palindrome if it is the same read forwards and backwards (for example 172271 is a palindrome, but 2320 is not.)
   How many $2k + 1$ digit numbers are not palindromes?

2. How many arrangements of MISSISSIPPI do not have consecutive I's?

3. Recall that a *derangement* of $\{1, 2, \ldots, n\}$ is a permutation with no fixed points. Let $D(n)$ denote the number of derangements of $\{1, 2, \ldots, n\}$. By convention $D(0) = 1$.

   (a) Directly calculate $D(1), D(2)$, and $D(3)$.

   (b) Give a combinatorial explanation for the following identity:

   $$n! = \sum_{k=0}^{n} \binom{n}{k} \cdot D(n-k),$$

   (that is, show that both sides above are counting the same thing.)

   (c) Use the identity from the previous part to compute $D(4)$.

4. Let $F(n)$ be the number of subsets of the set $\{1, 2, \ldots, n\}$ that contain no three consecutive integers. For example, $F(1) = 2$, $F(2) = 4$, and $F(3) = 7$.

   Find a recurrence satisfied by $F(n)$.

5. For a positive integer $n \geq 0$, show that

   $$\sum_{k=0}^{n} \frac{1}{k+1} \cdot \binom{n}{k} = \frac{2^{n+1} - 1}{n+1}.$$

   Hint: You may want to rewrite this equation somehow instead of trying to apply induction right away.

6. Suppose $n \geq 2$. We choose $n + 2$ numbers from the set $\{1, 2, \ldots, 3n\}$. Prove that there are always two among the chosen numbers whose difference is more than n but less than $2n$.

7. How many six letter words in the English alphabet either begin aa or end zz?