① $$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\cdot 2^k = \sum_{k=0}^{n}(-2)^k\binom{n}{k}\cdot 1^{n-k}$$

By Binomial Thm $\longrightarrow$ $$= (-2+1)^n$$

$$= (-1)^n;$$

The Statement is True.

② Each positive integer can be written in a unique way as a sum of of Fibonacci numbers.

The Statement is <u>false</u> as written.

For example, $5 = 2+3 = 3+1+1$.

If we impose that the Fibonacci numbers used be distinct, then the statement is true.

$n, k \in \mathbb{Z}_+$, $m < n$. Show

(5) $\binom{n}{k} \cdot \binom{k}{m} = \binom{n}{m} \cdot \binom{n-m}{k-m}$.

<u>Combinatorial argument</u>

Imagine out of n people we must form a committee of k people, then we form a subcommittee of m out of the k on the committee. The number of ways to do this is

$\binom{n}{k} \cdot \binom{k}{m}$,

This is the same as first choosing the m subcommittee members out of all n candidates then selecting the rest of the members to fill out the committee from the $n-m$ that remain. There are

$\binom{n}{m} \cdot \binom{n-m}{k-m}$

ways to do this.

Since the two expressions count the same thing, they are equal.

Algebra approach: expanding both $\binom{n}{k}$ and $\binom{k}{m}$ and rewriting will work as well.

Prove that for any $n > m$,

(6) $$\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}.$$

proof by induction on m. (induction runs from 0 to $n-2$).

The statement clearly holds for $m=0$.

Suppose that for some $m \geq 0$, the statement holds.

Then we have

$$\sum_{k=0}^{m+1} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m} + (-1)^{m+1} \binom{n}{m+1} =$$

$$= (-1)^{m+1} \left(\binom{n}{m+1} - \binom{n-1}{m} \right)$$

$$= (-1)^{m+1} \binom{n-1}{m+1}$$

By Pascal's identity.

Thus, the statement holds for each integer $m = 0, 1, \ldots, n-1$.

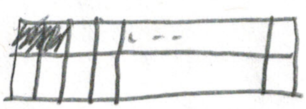
7 In how many ways can we cover a $2 \times N$ checkerboard by 2×1 dominoes?

If $N = 1$, there is clearly only one way.

If $N = 2$, There are 2 ways.

If $N = 3$, a vertical placement in the top right corner leaves a 2×2 square that still has to be filled. There are 2 ways to do this. A horizontal placement in the same corner leaves no choice for how to fill in the rest, so there are 3 total ways.

And so on. For general N,



A horizontal placement leaves a $2 \times (N-2)$ board to fill in.

A vertical placement leaves a $2 \times (N-1)$ board to fill in.

So the number of ways to cover the $2 \times N$ board is the sum of the number of ways to cover a $2 \times (N-2)$ board and the number of ways to cover a $2 \times (N-1)$ board.

Thus, the answer for N is the $N+1^{\text{Th}}$ Fibonacci Number.

⑧ Prove by induction that for any integer $n \geq 2$,

$$\binom{2n}{n} \leq 3 \cdot 2^{2n-3}.$$

$\binom{4}{2} = 6 \leq 6$, so the case $n=2$ holds.

Suppose that for some $n \geq 2$, we have

$\binom{2n}{n} \leq 3 \cdot 2^{2n-3}$. By Pascal's identity,

$$\binom{2(n+1)}{n+1} = \binom{2(n+1)-1}{n} + \binom{2(n+1)-1}{n+1} =$$

$$= \binom{2(n+1)-2}{n-1} + \binom{2(n+1)-2}{n} + \binom{2(n+1)-2}{n} + \binom{2(n+1)-2}{n+1}$$

$$= \binom{2n}{n-1} + \binom{2n}{n} + \binom{2n}{n} + \binom{2n}{n+1} \leq$$

$$\leq 4\binom{2n}{n} \leq 4 \cdot (3 \cdot 2^{2n-3}) = 3 \cdot 2^{2(n+1)-3},$$ as required.

Above we used Pascal's identity in two steps, and we used that the binomial coefficients are maximized "in the middle of Pascal's triangle.

Credit: This is Jon's Proof