Math 175: Combinatorics
Exam 1 Practice Problems

1 True/False: 3 Points Each

1. For any positive integer n,
$$\sum_{k=0}^{n}(-1)^k \cdot \binom{n}{k} \cdot 2^k = (-1)^n.$$
2. Every positive integer can be written in a unique way as a sum of Fibonacci numbers.

2 Problems: 10 Points Each

1. Prove that for $n \geq 1$,
$$\sum_{i=1}^{n}(i \cdot 2^i - 1) = (n-1)2^{n+1} - n + 2.$$
2. Suppose I play a game where I flip a coin n times, write down the number of heads, then flip a coin n times again and write down the number of heads in this second set of n flips. I win if the two numbers are the same and lose otherwise. Give a formula in terms of n for the probability that I win this game.
(Your final answer for this formula should not involve a sum.)

   (For example, if $n = 4$ I win if the first four flips are HHTH and the second four flips are THHH, but I lose if the first four flips are HTTH and the second four are HHHT.)
3. Two systems are proposed for the Irvine Lottery. In the first system, players select six *different* numbers from $\{1, 2, \ldots, 50\}$ for their tickets. Two tickets are considered the same if they contain the same numbers in different orders.

   In the second system, players select six numbers from $\{1, 2, \ldots, 45\}$ *and* may select any number as many times as they want. In a drawing for this second system each ball selected in the lottery drawing is replaced before another ball is selected.

   Two tickets are considered the same if they contain the same multiset of numbers in different orders. For example, $\{3, 3, 1, 1, 1, 2\}$ and $\{1, 1, 1, 3, 2, 3\}$ are the same ticket and both are winners if 2 is the first ball drawn, the next two balls are 3, and the last three balls are 1.

   How many different tickets are there for each system?
4. A palindrome is a word that is spelled the same backwards and forwards- for example, KAYAK is one, but APPLE is not.
   (a) How many five letter palindromes are there using the English alphabet?
   (b) How many six letter palindromes are there using the English alphabet?

5. Prove that if n and k are nonnegative integers and m is an integer with $m \leq n$, then

$$\binom{n}{k} \cdot \binom{k}{m} = \binom{n}{m} \cdot \binom{n-m}{k-m}.$$

6. Prove that for any positive integers $n > m$,

$$\sum_{k=0}^{m} (-1)^k \binom{n}{k} = (-1)^m \binom{n-1}{m}.$$

7. In how many ways can you cover a $2 \times N$ chessboard with 2×1 dominoes?

8. Prove by induction that for any positive integer $n \geq 2$,

$$\binom{2n}{n} \leq 3 \cdot 2^{2n-3}.$$