175

L9 Binomial Identities

3.6 LVP What is the sum of the squares of the elements in each row of Pascal's Triangle?

n			
0	1	1^2	$= 1$
1	1 1	1^2+1^2	$= \boxed{2}$
2	1 $\boxed{2}$ 1	$1^2+2^2+1^2$	$= 6$
3	1 3 3 1	$1^2+3^2+3^2+1^2$	$= 20$
4	1 4 6 4 1	$1^2+4^2+6^2+4^2+1^2$	$= 70$
5	1 5 10 10 5 1		
6	1 6 15 20 15 6 1		

Conjecture:

$$\sum_{k=0}^{n} \binom{n}{k}^2 = \binom{2n}{n}$$

Pf We argue that the LHS and RHS are different ways of counting the same things.

RHS = # ways to choose n objects from 2n objects

$\stackrel{eg}{=}$ # n-subsets of $\{1, 2, \ldots, n, n+1, \ldots 2n\}$

Consider such a subset. Count the # elements that come from $\{1, 2, \ldots, n\}$.

Let this number be k. Then the remaining $n-k$ elements must come from $\{n+1, \ldots, 2n\}$.
How many subsets of $\{1, 2, \ldots, 2n\}$ are there containing a particular k-subset from $\{1, 2, \ldots, n\}$?
. . . The answer is the # ways of choosing $n-k$ elements from $\{n+1, n+2, \ldots, 2n\}$, which is $\binom{n}{n-k}$.

But there are $\binom{n}{k}$ ways to choose a subset of $\{1, \ldots, 2n\}$ in which k elements come from $\{1, \ldots, n\}$. Thus there are

$$\binom{n}{k}\binom{n}{n-k}$$

subsets of $\{1, 2, \ldots, 2n\}$ in which k elements come from 1st half and $n-k$ elements come from 2nd half. Since subsets with different k cannot possibly coincide, we have:

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}\binom{n}{n-k}$$

$$= \sum_{k=0}^{n} \binom{n}{k}^2 \quad \text{(symmetry of Pascals } \Delta\text{)}$$

An equivalent but different way of saying this is to imagine that you're flipping a coin $2n$ times. How can you get exactly n heads? Answer $\binom{2n}{n}$. Suppose you get k heads in the first n tosses. How many ways could this occur: $\binom{n}{k}$. You must then get $n-k$ heads in the remaining n tosses. How many ways: $\binom{n}{n-k}$. Total # ways to get k heads in the first n tosses and $n-k$ in the remainder is

$$\binom{n}{k}\binom{n}{n-k}$$

Since k could be anywhere from 0 to n, we have

$$\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2.$$

A generalization of this result is Vandermonde's identity:

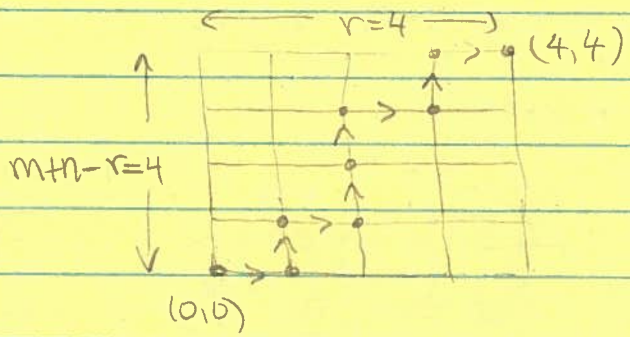
$\dfrac{\binom{m}{k}\binom{n}{r-k}}{\binom{m+n}{r}}$ = hyper-geometric prob. distn

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

There are three ways to prove this (1) as above (2) binomial thm (later) and

(3) geometrically:

Pf:

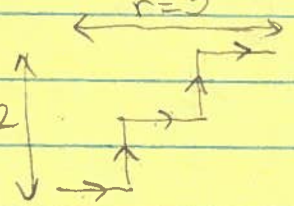


only "up-" or "right-" moves allowed

How many ways to go from $(0,0)$ to $(r, m+n-r)$?

In any given path, there are r right moves and $m+n-r$ up moves. $\Longrightarrow$ The total path length is # right moves plus # left moves, which is $r + [m+n-r] = m+n$.

Each path can be viewed as an assignation of r right arrows to $m+n$ dots:



$\equiv \underbrace{\underline{\rightarrow}\ \underline{\uparrow}\ \underline{\rightarrow}\ \underline{\uparrow}\ \underline{\rightarrow}}_{m+n=5} \equiv \underline{\rightarrow}\ \underline{\ }\ \underline{\rightarrow}\ \underline{\ }\ \underline{\rightarrow}$

$\underline{\uparrow}\ \underline{\rightarrow}\ \underline{\rightarrow}\ \underline{\uparrow}\ \underline{\rightarrow} = \underline{\ }\ \underline{\rightarrow}\ \underline{\rightarrow}\ \underline{\ }\ \underline{\rightarrow}$

Since right arrows are indistinguishable, the # distinguishable ways to place them is:

$$\binom{m+n}{r},$$

which is therefore the # paths.

We now count the paths in another way:

Choose a point on the grid with coordinates $(k, m-k)$. By the same argument as above, there are

$$\binom{m}{k}$$

paths from $(0,0)$ to $(k, m-k)$ since the path length is $k+(m-k)=m$ and k right moves must take place.

Similarly there are $\binom{n}{r-k}$

paths from $(k, m-k)$ to $(r, m+n-r)$ since path length is

$$\underbrace{r-k}_{\# \rightarrow} + \underbrace{(m+n-r)-(m-k)}_{\# \uparrow} = n$$

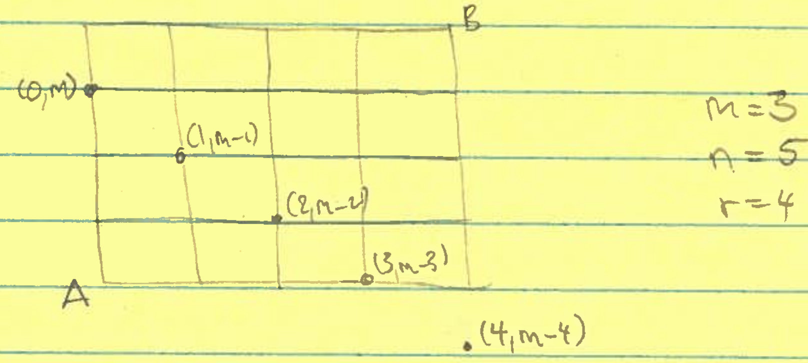
and # right moves is $r-k$.

Thus # paths that pass through $(k, m-k)$

is

$$\binom{m}{k}\binom{n}{r-k}.$$

Since the set of possible intermediate pts looks like:



a path from A to B has no choice except to pass thru' one of them. Thus total # paths is:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}$$

∎

A shorter way to obtain $\sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k}$ is:

Partition the paths into those in which there are k right moves in the first m steps and $r-k$ right moves in the remaining n steps:

$$\underbrace{\overset{\longleftarrow\, m=3\, \longrightarrow}{\underbrace{\overset{\rightarrow}{\_}\;\;\_\;\;\overset{\rightarrow}{\_}}_{\#(\rightarrow)=k=2}}\;\;\overset{\longleftarrow\, n=2\, \longrightarrow}{\underbrace{\_\;\;\overset{\rightarrow}{\_}}_{\#(\rightarrow)=r-k=1}}}_{\#(\rightarrow)=r=3}.$$

$\#$ such paths $= \binom{m}{k}\binom{n}{r-k}$

$\Rightarrow$ total $\#$ paths $= \sum_{k=0}^{r}\binom{m}{k}\binom{n}{r-k}$

[Thanks to Christian Walker for pointing this out].