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L8 — More Fibonacci Facts.

4.3 LVP — How big are the Fibonacci Numbers?

Theorem $$F_n = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right] \quad n \geq 0.$$

Pf Argue by induction on n.

base cases:

$n=0$: $F_0 = \frac{1}{\sqrt{5}}[1-1] = 0$ ✓

$n=1$: $F_1 = \frac{1}{\sqrt{5}}\left[\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}\right]$

$= \frac{1}{\sqrt{5}}[\sqrt{5}] = 1$ ✓

Suppose theorem is true for $n, n-1$.

Goal: prove true for $n+1$. Now:

$$F_{n+1} = F_n + F_{n-1}$$

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right]$$

$$+ \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n-1}\right]$$ (by induction hypothesis)

$$= \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1}\frac{2}{1+\sqrt{5}}\left[1+\frac{2}{1+\sqrt{5}}\right]\right.$$

$$\left. - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\frac{2}{1-\sqrt{5}}\left[1+\frac{2}{1-\sqrt{5}}\right]\right]$$

But:

$$\left[1+\frac{2}{1+\sqrt{5}}\right] = \frac{1+\sqrt{5}+2}{1+\sqrt{5}} = \frac{3+\sqrt{5}}{1+\sqrt{5}}$$

$$= \frac{1+\sqrt{5}}{2} \cdot \frac{2}{1+\sqrt{5}} \cdot \frac{3+\sqrt{5}}{1+\sqrt{5}}$$

$$= \frac{1+\sqrt{5}}{2} \cdot \frac{6+2\sqrt{5}}{\underbrace{1+2\sqrt{5}+5}_{1}}$$

$$= \frac{1+\sqrt{5}}{2}$$

Similarly

$$\left[1+\frac{2}{1-\sqrt{5}}\right] = \frac{1-\sqrt{5}+2}{1-\sqrt{5}} = \frac{3-\sqrt{5}}{1-\sqrt{5}}$$

$$= \frac{1-\sqrt{5}}{2} \cdot \frac{2}{1-\sqrt{5}} \cdot \frac{3-\sqrt{5}}{1-\sqrt{5}}$$

$$= \frac{1-\sqrt{5}}{2} \cdot \frac{6-2\sqrt{5}}{\underbrace{1-2\sqrt{5}+5}_{1}}$$

$$= \frac{1-\sqrt{5}}{2}$$

Thus

$$F_{n+1} = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{n+1}\right]$$

Note: for n large $\left(\frac{1+\sqrt{5}}{2}\right)^n$ is much larger in mag. than $\left(\frac{1-\sqrt{5}}{2}\right)^n$ since $\frac{1+\sqrt{5}}{2} > 1$ while $\left|\frac{1-\sqrt{5}}{2}\right| < 1$.

Thus:

$$F_n \underset{n \gg}{\cong} \frac{1}{\sqrt{5}} \left(\frac{1+\sqrt{5}}{2}\right)^n$$

The number $\frac{1+\sqrt{5}}{2}$ is called the Golden Ratio and denoted φ. Thus:

$$F_n \underset{n \gg 1}{\cong} \frac{1}{\sqrt{5}} \varphi^n.$$

Recall that the first few Fibo numbers are:

$$0, 1, 1, 2, 3, 5, 8, \ldots$$

Can we break a given number into a sum of Fibo numbers? Some test cases:

$$\begin{aligned} 9 &= 5 + 2 + 1 + 1 & (1) \\ &= 5 + 3 + 1 & (2) \\ &= 8 + 1 & (3) \end{aligned}$$

In (1) two Fibo numbers are repeated. In (2) they are distinct, but so they are also in (3).

Claim Every positive integer can be written as a sum of different Fibonacci numbers.

Outline of Pf Argue by induction.

Base case $n=1$: obvious since 1 is a Fibonacci number

Induction hypothesis: Claim is true for every positive integer $\leq n$.

WTS: $n+1$ can be written as a sum of different Fibonacci numbers.

Trick: use a "greedy" algorithm for constructing a decomposition:

Let F_k be the largest Fibo. number that "fits into" $n+1$. Then

$$n+1 = F_k + \underbrace{(n+1)-F_k}_{\substack{\text{positive integer } \leq n \\ \Rightarrow \text{use induction} \\ \text{hypothesis.}}}$$

We need to rule out the possibility that

F_k appears in the decomposition of $(n+1) - F_k$, ie. WTS

$$(n+1) - F_k < F_k$$

$$\Rightarrow \quad n+1 < 2F_k$$

$$\Rightarrow \quad F_k > \frac{n+1}{2} \qquad (*)$$

When this is the case, we can "recur backwards" to convince ourselves that no two Fibo numbers in the decomposition of n+1 are the same.

Hint: use the recurrence relation that defines the Fibonacci numbers,

$$F_{k+1} = F_k + F_{k-1}$$

to prove (*).

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A stronger version of the claim is Zeckendorf's Theorem (see wikipedia entry): the decomposition is unique if we consider only decompositions that don't include consecutive Fibo numbers, eg: $9 = 8 + 1$.

These decompositions can be obtained by the greedy algorithm of always choosing the largest Fibo that "fits".

Q. How many ways are there to climb n steps taking 1 or 2 steps at a time?

A. Numerical exploration to try and spot a pattern:

# steps n	ways to climb	# ways
1	1	1
2	1+1, 2	2
3	1+1+1, 1+2, 2+1	3
4	1+1+1+1, 1+1+2, 1+2+1, 2+1+1, 2+2	5

The pattern here is that each way to climb n steps is a composition of n, using only the integers 1 or 2.

Let $g(n)$ = the number of such compositions.

<u>Claim</u>: $g(n+1) = g(n) + g(n-1)$.

Pf Trick is to show that LHS and RHS count the same things.

By definition LHS counts # of compositions of $n+1$ into 1^s and 2^s.

Each such composition ends w/ a 1 or a 2.

If it ends w/ a 1, then we can remove the 1 to obtain a composition of n. Conversely, to any composition of n we may add a 1 to obtain a composition of $n+1$ ending with a 1.

Similarly we may construct a bijection between compositions of $n+1$ that end with a 2 and compositions of $n-1$.

Since compositions of $n+1$ end either with a 1 or a 2 (and not both), we have:

$$g(n+1) = \#\{\text{compositions of } n+1 \text{ that end with } 1\} + \#\{\text{compositions of } n+1 \text{ that end with } 1\}$$

$$= g(n) + g(n-1).$$

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Since this is the recurrence relation satisfied by the Fibonacci sequence, we may conclude that #ways to climb the stairs is one of the Fibonacci numbers. But which one?

Revisit table:

#stairs n	# ways	Fibonacci
1.	1	F_2
2	2	F_3
3.	3	F_4
4	5	F_5

Thus: # ways to climb n stairs in leaps of 1 or 2 is F_{n+1}.

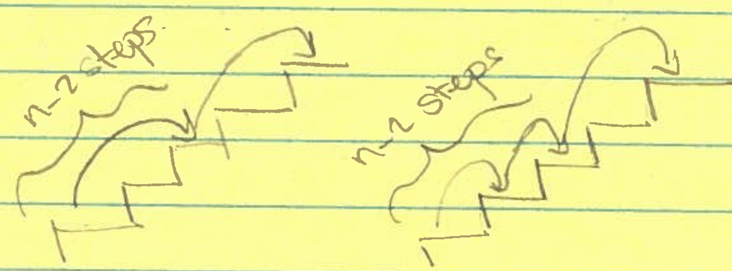
By the way, there is another way to view the proof we presented:

Break the ways up into two disjoint sets by "conditioning" on the last "leap" the person takes.

last leap = 1 step.

#ways = $g(n-1)$

last leap = 2 steps



#ways = $g(n-2)$

The result follows since a person arrives @ the top taking either 1 or 2 steps as the final "leap".

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