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L7

3.6 LVP

Hockey-stick identity

Consider the following "hockey stick" of numbers:

$$
\begin{array}{ccccccccc}
 & & & & \binom{0}{0} & & & & \\
 & & & \binom{1}{0} & & \binom{1}{1} & & & \\
 & & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & & \\
 & \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{4} & \\
\binom{4}{0} & & \binom{4}{1} & & \binom{4}{2} & & \binom{4}{3} & & \binom{4}{4}
\end{array}
\quad = \quad
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & & 1 & & & \\
 & & 1 & & 2 & & 1 & & \\
 & 1 & & 3 & & 3 & & 1 & \\
1 & & 4 & & 6 & & 4 & & 1
\end{array}
$$

Evidently the sum of the entries along the "handle" equals the entry in the "blade":

$$1 + 2 + 3 = 6$$

or

$$\binom{1}{1} + \binom{2}{1} + \binom{3}{1} = \binom{4}{2}$$

This works for other hockey sticks too:

$$
\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & & 1 & & & & \\
 & & & 1 & & 2 & & 1 & & & \\
 & & 1 & & 3 & & 3 & & 1 & & \\
 & 1 & & 4 & & 6 & & 4 & & 1 & \\
1 & & 5 & & 10 & & 10 & & 5 & & 1
\end{array}
$$

$$\binom{1}{0} + \binom{2}{1} + \binom{3}{2} + \binom{4}{3} = \binom{5}{3}$$

Proposition (Hockey-stick)

$$\sum_{j=0}^{k} \binom{n+j}{j} = \binom{n+k+1}{k} \qquad (*)$$

Proof by induction

For any fixed n, we argue by induction on k:

Base case $k=0$:

$$\sum_{j=0}^{0} \binom{n+j}{j} = \binom{n}{0} = \binom{n+1}{0} = \binom{n+k+1}{k} \checkmark$$

Suppose $(*)$ is true for k.

Goal is to show that:

$$\sum_{j=0}^{k+1} \binom{n+j}{j} = \binom{n+(k+1)+1}{k+1} \qquad (**)$$

Now

$$\sum_{j=0}^{k+1} \binom{n+j}{j} = \sum_{j=0}^{k} \binom{n+j}{j} + \binom{n+k+1}{k+1}$$

$$= \binom{n+k+1}{k} + \binom{n+k+1}{k+1}$$

$$= \binom{(n+k+1)+1}{k+1} \qquad \text{(Pascals identity/rule p9, L6)}$$

which is $(**)$. $\boxtimes$

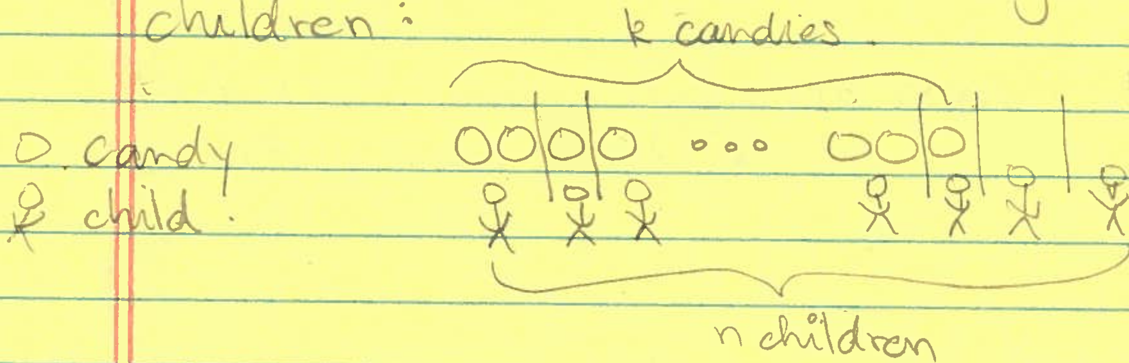
Combinatorial proof

Recall from L4 (and p12 of L5):

\# k-multisets of an n-set is:

$$\binom{k+n-1}{n-1} \qquad (1)$$

An example is distributing k indistinguishable candies to n distinguishable children:



Q In how many of these distributions does the "first" child receive i candies ($i = 0, 1, \dots, k$)

A. If the first child receives i candies, then the remaining $n-1$ children receive $k-i$ candies. Using (1), this can happen in

$$\binom{(k-i)+(n-1)-1}{(n-1)-1} \quad \text{ways}$$

Since each distribution is characterized by the first child receiving i candies for some i in the range 0 to k, we must have that:

$$\binom{k+n-1}{n-1} = \sum_{i=0}^{k} \binom{(k-i)+(n-1)-1}{(n-1)-1}$$

$$= \sum_{i=0}^{k} \binom{\overbrace{k-i}^{j}+n-2}{n-2}$$

$$= \sum_{j=0}^{k} \binom{j+n-2}{n-2}$$

ie. $$\binom{k+n-1}{n-1} = \sum_{j=0}^{k} \binom{n-2+j}{j} \qquad (2)$$

Putting $n' = n-2$ in (2) yields:

$$\binom{k+(n'+1)}{(n'+1)} = \sum_{j=0}^{k} \binom{n'+j}{j}$$

which is $(*)$ since LHS is

$$\binom{(n'+1)+k}{k} = \binom{n'+k+1}{k}.$$

Fibonacci Numbers

4.1 LVP

Left-justify the rows of Pascals Triangle:

$$
\begin{array}{ccccccccccccc}
 & & & & & & 1 & & & & & & \\
 & & & & & 1 & & 1 & & & & & \\
 & & & & 1 & & 2 & & 1 & & & & \\
 & & & 1 & & 3 & & 3 & & 1 & & & \\
 & & 1 & & 4 & & 6 & & 4 & & 1 & & \\
 & 1 & & 5 & & 10 & & 10 & & 5 & & 1 & \\
1 & & 6 & & 15 & & 20 & & 15 & & 6 & & 1
\end{array}
\longrightarrow
\begin{array}{lllllll}
1 & & & & & & \\
1 & 1 & & & & & \\
1 & 2 & 1 & & & & \\
1 & 3 & 3 & 1 & & & \\
1 & 4 & 6 & 4 & 1 & & \\
1 & 5 & 10 & 10 & 5 & 1 & \\
1 & 6 & 15 & 20 & 15 & 6 & 1
\end{array}
$$

1, 1, 2, 3, 5, 8, 13 (written at the ends of the diagonals)

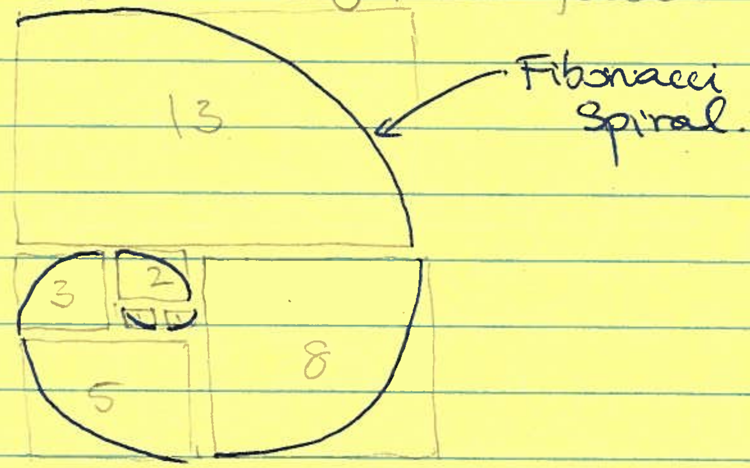
The sum of the diagonal elements in left-justified table are known as the Fibonacci numbers.

$$1, 1, 2, 3, 5, 8, 13, \ldots$$

If you look at those for a while you'll notice that every number after the first two is the sum of the two preceding ones. If we place a zero at the beginning of the sequence, we have

$$F_0 = 0$$
$$F_1 = 1$$
$$F_n = F_{n-1} + F_{n-2} \qquad n \geq 2$$

A nice visualization of this fact is



Proposition (Fibonacci Identity)

$$\binom{n}{0} + \binom{n-1}{1} + \cdots + \binom{n-k}{k} = F_{n+1}$$

where

$$k = \lfloor n/2 \rfloor$$

Pf Argue by induction on n.

base cases.

$n=0$: $\binom{0}{0} = 1 = F_1$ ✓

$n=1$ $\binom{1}{0} = 1 = F_2$ ✓

Suppose true for $n-2, n-1$ $(n \geq 3)$

Want to prove true for n. Consider:

$$\binom{n}{0}+\binom{n-1}{1}+\binom{n-2}{2}+\cdots+\binom{n-k}{k} \qquad (3)$$

But:

$$\binom{m}{r} = \binom{m-1}{r} + \binom{m-1}{r-1} \qquad \text{(Pascal's Rule)}$$

Thus (3) can be written:

$$\binom{n-1}{0} + \left[\binom{n-2}{1}+\binom{n-2}{0}\right] + \left[\binom{n-3}{2}+\binom{n-3}{1}\right] + \cdots$$
$$+ \left[\binom{n-k-1}{k}+\binom{n-k-1}{k-1}\right]$$

Reorganize:

$$\left[\binom{n-1}{0}+\binom{n-2}{1}+\binom{n-3}{2}+\cdots+\binom{n-k-1}{k}\right]$$

$$+\left[\binom{n-2}{0}+\binom{n-3}{1}+\binom{n-4}{2}+\cdots+\binom{n-k-1}{k-1}\right]$$

Now assume n is odd. Then:

$$\left[\binom{n-1}{0}+\binom{(n-1)-1}{1}+\binom{(n-1)-2}{2}+\cdots+\binom{(n-1)-k'}{k'}\right]$$

$$\left[\binom{n-2}{0}+\binom{(n-2)-1}{1}+\binom{(n-2)-2}{2}+\cdots+\binom{(n-2)-k''}{k''}\right] \qquad (4)$$

where

$$k' = \left\lfloor \frac{n-1}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor = k$$

$$k'' = \left\lfloor \frac{n-2}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor - 1 = k-1$$

[For example:

n	3	4	5	6	7	8
$\frac{n}{2}$	1.5	2	2.5	3	3.5	4
$\lfloor \frac{n}{2} \rfloor$	1	2	2	3	3	4

]

Finally (4) ⇒

$$F_{(n-1)+1} + F_{(n-2)+1}$$ [induction hypothesis twice]

$$= F_n + F_{n-1}$$

$$= F_{n+1}$$

which is the proposition for n odd. Similar derivation exists for n even.

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