L6 **Birthday Paradox : non-uniform birthdates**
25 LVP (continued)

EX 2 people. 3 birthdays $\{A, B, C\}$ w/ prob of occurrence $\{p_1, p_2, p_3\}$. Show that probability of birthday collision is smallest when birthdays are uniform, $p_1 = p_2 = p_3$.

Sol^n Sample space:

AA	BA	CA
AB	BB	CB
AC	BC	CC.

$$P(\text{collision}) = p_1^2 + p_2^2 + p_3^3$$

$$= p_1^2 + p_2^2 + (1 - p_1 - p_2)^2$$

$$= f(p_1, p_2).$$

$$\vec{\nabla} f = \left\langle \frac{\partial f}{\partial p_1}, \frac{\partial f}{\partial p_2} \right\rangle$$

Consider:

$$\frac{\partial f}{\partial p_1} = 2p_1 + 2(1 - p_1 - p_2)(-1)$$

$$= 2\left[2p_1 + p_2 - 1\right]$$

$$\frac{\partial f}{\partial p_1} = 0 \Rightarrow 2p_1 + p_2 = 1 \qquad (1)$$

Similarly

$$\frac{\partial f}{\partial p_2} = 0 \Rightarrow p_1 + 2p_2 = 1 \qquad (2)$$

(1) & (2) $\Rightarrow$ $p_1 = p_2 = 1/3$. $(= p_3)$.

We can see that this is a minimum of $f(p_1, p_2)$ by examining, eg

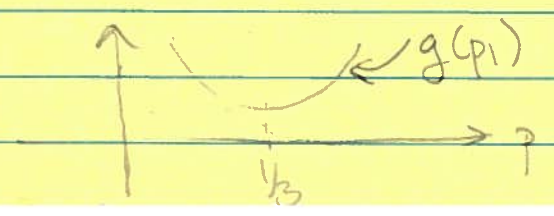
$$g(p_1) = f(p_1, 1/3)$$

$$\Rightarrow g'(p_1) = \frac{\partial f}{\partial p_1} = 2[2p_1 + (\tfrac{1}{3}) - 1].$$

$$= 2(2p_1 - 2/3)$$

$$= 4(p_1 - 1/3)$$

$$\Rightarrow g'(p_1) = \begin{cases} < 0 & p_1 < 1/3 \\ 0 & p_1 = 1/3 \\ > 0 & p_1 > 1/3 \end{cases}$$

$\Rightarrow$ 

$\Rightarrow$ $p_1 = p_2 = p_3 = 1/3$
$\rightarrow$ minimum of
of $f(p_1, p_2)$

Induction

§2.1 LVP

Suppose we have some property about positive numbers. Suppose further that we know that:

1. 1 has this property (base case)
2. whenever $n-1$ has this property, so does n $(n>1)$

Then the principle of induction says that every positive n has this property:

EX $$\sum_{j=0}^{n} 2^j = 2^{n+1} - 1 \qquad (*)$$

Pf Check this for a base case, eg $n=0$

$$2^0 = 2^{0+1} - 1 \quad \checkmark$$

Suppose true for n. This is often called the "induction hypothesis". Use this to prove $(*)$ for $n+1$:

$$\sum_{j=0}^{n+1} 2^j = \sum_{j=0}^{n} 2^j + 2^{n+1}$$

$$= 2^{n+1} - 1 + 2^{n+1}$$

$$= 2 \cdot 2^{n+1} - 1$$

$$= 2^{n+2} - 1$$

$$= 2^{(n+1)+1} - 1$$

which is (*) with n replaced by $n+1$.

Ex $$\sum_{j=1}^{n} j = \frac{n(n+1)}{2} \qquad (**)$$

Pf Check for $n=1$: $1 = \frac{1(1+1)}{2}$ ✓

Induction hypothesis: $\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$

Prove for $n+1$:

$$\sum_{j=1}^{n+1} j = \sum_{j=1}^{n} j + (n+1) = \frac{n(n+1)}{2} + (n+1)$$

$$= \frac{(n+1)}{2}\left[n + 2\right] = \frac{(n+1)((n+1)+1)}{2}$$

which is (**) w/ $n \to n+1$.

Ex $$\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6} \qquad (***)$$

Pf Check for $n=1$: $1^2 = \frac{1(1+1)(2\cdot1+1)}{6}$ ✓

Induction hypothesis: $\sum_{j=1}^{n} j^2 = \frac{n(n+1)(2n+1)}{6}$

Prove for $n+1$:

$$\sum_{j=1}^{n+1} j^2 = \sum_{j=1}^{n} j^2 + (n+1)^2 = \frac{n(n+1)(2n+1)}{6} + (n+1)^2$$

$$= \frac{(n+1)}{6}\left[n(2n+1) + 6(n+1)\right]$$

$$= \frac{(n+1)}{6}\left[\begin{matrix} 2n^2 + n \\ + 6n + 6 \end{matrix}\right]$$

$$= \frac{n+1}{6}\left[2n^2 + 7n + 6\right]$$

$$= \frac{n+1}{6}(n+2)(2n+3)$$

$$= \frac{(n+1)((n+1)+1)(2(n+1)+1)}{6}$$

which is $(*)$ for $n+1$

Recall: $$\sum_{j=1}^{n} j = \frac{n(n+1)}{2} \qquad (**)$$

Another way to establish this is to view both sides as counting the same thing in two different ways.

$$RHS = \frac{(n+1)n}{2 \cdot 1} = \binom{n+1}{2}$$

$= \#$ways to choose 2 items from $\{1, 2, \dots, n+1\}$

$$= \#\{(1,2), (1,3), (1,4), \dots, (1,n+1)\}$$

$$+ \quad \#\{(2,3), (2,4), \dots, (2,n+1)\}$$

$$+ \quad \#\{(3,4), \dots, (3,n+1)\}$$

$$+ \quad \#\{(n,n+1)\}$$

$$= n + (n-1) + (n-2) + \dots + 1$$

$= LHS$.

In L5, we proved that the number of (strong) compositions of n is 2^{n-1}. We now present an inductive proof.

BASE CASE: $n=1$: # compositions $= 1 = 2^0 = 2^{1-1}$. ✓

INDUCTION HYPOTHESIS: # compositions of m is 2^{m-1}, $1 \le m \le n$.

INDUCTIVE STEP: Prove : # compositions of $n+1$ is $2^{(n+1)-1} = 2^n$.

We do this by answering a number of questions:

Q. How many compositions of $n+1$ have 1 as its "right-most" part?, eg $2+1$ but not $1+2$

A. Take that part away and get a composition of n. But there are 2^{n-1} compositions of n (induction hypothesis). Thus there are 2^{n-1} compositions of $n+1$ w/ 1 as right-most part.

Q How many have k as the right-most part?

A Take that part away and get a composition of $n+1-k$, which has $2^{(n+1-k)-1} = 2^{n-k}$ compositions. (induction hypothesis)

Q How many have last part between 1 and n, inclusive?

A.
$$\underset{(k=1)}{2^{n-1}} + \underset{(k=2)}{2^{n-2}} + \cdots + \underset{(k=n)}{2^{0}} =$$

$$= \sum_{j=0}^{n-1} 2^j = 2^{(n-1)+1} - 1 \quad \text{by } (*)$$

$$= 2^n - 1.$$

Since this is the number of all compositions of $n+1$ except itself, we may conclude that $n+1$ has $(2^n-1)+1 = 2^n$ compositions.

Pascals Triangle

3.5 LVP

Recall:

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

$$= \text{binomial coefficient}$$

BTW:

$$\binom{n}{n-k} = \frac{n!}{(n-(n-k))!\,(n-k)!} = \frac{n!}{(n-k)!\,k!}$$

Thus:

$$\binom{n}{k} = \binom{n}{n-k}$$

We arrange all binomial coefficients into a triangular scheme:

$$\begin{array}{ccccccc} & & & \binom{0}{0} & & & \\ & & \binom{1}{0} & & \binom{1}{1} & & \\ & \binom{2}{0} & & \binom{2}{1} & & \binom{2}{2} & \\ \binom{3}{0} & & \binom{3}{1} & & \binom{3}{2} & & \binom{3}{3} \end{array} \quad \text{etc.}$$

This is called Pascal's Triangle. Replacing each binomial coeff. by its value, we get:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 2 & & 1 & & \\ & 1 & & 3 & & 3 & & 1 & \\ 1 & & 4 & & 6 & & 4 & & 1 \end{array}$$

Pascals Triangle is symmetric about the vertical line through it's apex because

$$\binom{n}{k} = \binom{n}{n-k} \quad \text{(see above).}$$

If you examine the numbers in the triangle, then it appears that

3.6 LVP
$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

Pf 1 WTS:
$$\frac{n!}{(n-k)!\,k!} = \frac{(n-1)!}{(n-k)!\,(k-1)!} + \frac{(n-1)!}{(n-1-k)!\,k!}$$

Divide both sides by

$$\frac{(n-1)!}{(n-1-k)!\,(k-1)!}$$

to get:

$$\frac{n}{(n-k)k} = \frac{1}{n-k} + \frac{1}{k}$$

But:

$$\text{RHS} = \frac{k + (n-k)}{(n-k)k}$$

$$= \frac{n}{(n-k)k} = \text{LHS}$$

⊠

Pf 2 Show that both sides count the same thing in two different ways.

(a) $\binom{n}{k}$ = # ways of choosing k elements from $\{1, 2, \ldots, n\}$

(b) $\binom{n-1}{k-1}$ = # ways of choosing the element n and k-1 other elements of $\{1, 2, \ldots, n-1\}$.

(c) $\binom{n-1}{k}$ = # ways of choosing k elements from $\{1, 2, \ldots, n-1\}$.

Since any set counted in (a) is either one of the sets counted in (b) or (c), we have

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

$\boxtimes$