Six Ways to Count the Number of Integer Compositions

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Counting problems are at the core of the field of study that we call "Combinatorics". Basic principles and techniques of Combinatorics familiar to any undergraduate student, and often to an advanced high school student, include basic laws of sum and product, counting distributions, permutations and combinations, the principle of Induction, and the principle of Inclusion & Exclusion. A useful method often employed to count the number of objects in a set is to place the set in one-to-one correspondence with another set whose size is more easily determined. The problem of counting the number of compositions of a positive integer is standard, and can be found in some form in several books that deal with basic combinatorial methods. We illuminate this problem by touching upon many different basic counting methods that may be employed to solve this problem.

Let n and k be positive integers. A `composition` of n into k (positive) parts is an ordered k-tuple $(x_1, \ldots, x_k)$ with each $x_i \in \mathbb{N}$ and $x_1 + \cdots + x_k = n$. The x_i's are the `parts` of the composition. Thus $(1, 2, 2, 3)$ is a composition of 8 into 4 parts. If we denote by $p_k^\star(n)$ the number of compositions of n into k parts, it is a standard result in combinatorics that

$$p_k^\star(n) = \left|\Big\{(x_1, \ldots, x_k) : x_1 + \cdots + x_k = n, x_i \geq 1\Big\}\right| = \binom{n-1}{k-1}. \tag{1}$$

It is worth recalling an ingenious method to solve this (and other similar) problems. Place n dashes on a line, with adjacent dashes separated by blank spaces. Note that there are $n-1$ blank spaces. Choose $k-1$ of these $n-1$ blank spaces and fill them with $k-1$ bars; this can be done in $\binom{n-1}{k-1}$ ways. Each of these ways breaks the n dashes into k nonempty batches, thus providing a (unique) composition of n into k parts. To see this, the figure below illustrates the example of $(1, 2, 2, 3)$ as a composition of 8 into 4 parts by placing bars into spaces 1, 3, and 5.

$$— {\scriptstyle 1} — {\scriptstyle 2} — {\scriptstyle 3} — {\scriptstyle 4} — {\scriptstyle 5} — {\scriptstyle 6} — {\scriptstyle 7} — \quad \longrightarrow \quad — \big| — \; — \big| — \; — \big| — \; — \; —$$

Figure 2: $(1, 2, 2, 3)$ as a composition of 8 into 4 parts

The selection of $k-1$ of the $n-1$ blank spaces to be filled with bars is associated in a one-to-one manner with the composition of n into k parts. Thus there are $\binom{n-1}{k-1}$ compositions of n into k parts.

A `composition` of n is a composition of n into k parts, with no restriction on the number of parts. We list below the number of compositions of n.

n	1	2	3	4	5	6	7	8	9	10
$p^\star(n)$	1	2	4	8	16	32	64	128	256	512

The pattern is unmistakeable, and suggests a simple combinatorial explanation. Indeed, if $p^\star(n)$ denotes the number of compositions of n, then

$$p^\star(n) = \sum_{k\geq 1} p_k^\star(n) = \sum_{k=1}^{n} \binom{n-1}{k-1} = \sum_{k=0}^{n-1} \binom{n-1}{k} = 2^{n-1}. \tag{2}$$

The method of placing bars between dashes to obtain a formula for $p_k^\star(n)$ can be easily adapted to also obtain the result in (2). In fact, since we have no restriction on the number of bars to place between the dashes, there are exactly 2 choices (whether or not to place a bar) between each pair of adjacent dashes. Each of these choices can again be associated in a one-to-one manner with the compositions of n, leading to 2^{n-1} compositions of n.

The simplicity of formula (2) for $p^\star(n)$ leads us to believe that there may be other ways to obtain this result by using other basic combinatorial methods. Underlying this is the understanding that any two finite sets of the same size must be in one-to-one correspondence, and it is this correspondence that we seek.

Induction. One of the first thoughts that cross our minds when we need to prove a formula that applies to the set of positive integers is to apply the method of mathematical induction. Note that $p^\star(1) = 2^0 = 1$. Assume that $p^\star(m) = 2^{m-1}$ for $1 \leq m \leq n-1$. Each composition of n begins with a k for some k with $1 \leq k \leq n$. Since there are $p^\star(n-k)$ compositions of n with first part k, $1 \leq k < n$, and one with first part n, using induction hypothesis we get

$$p^\star(n) = 1 + \sum_{k=1}^{n-1} p^\star(n-k) = (2^{n-2} + 2^{n-3} + \cdots + 2 + 1) + 1 = 2^{n-1}.$$

Recurrence. One of the simplest, yet powerful, methods to resolve a combinatorial problem is to find a recurrence equation satisfied by the function that solves the problem, then use standard methods to solve the recurrence. Recurrences are not easy to resolve in general, yet the sequence $\{p^\star(n)\}_{n\geq 1}$ looks promising given the sequence it represents.

The set of compositions of n with first part 1 is in one-to-one correspondence with the set of compositions of $n-1$ via $(1, a_2, a_3, \ldots, a_k) \leftrightarrow (a_2, a_3, \ldots, a_k)$. The set of compositions of n with first part $m > 1$ is in one-to-one correspondence with the set of compositions of $n-1$ with first part $m-1$ via $(a_1, a_2, \ldots, a_k) \leftrightarrow (a_1 - 1, a_2, \ldots, a_k)$. The latter set is the set of all compositions of $n-1$. Hence $p^\star(n) = 2 \cdot p^\star(n-1)$ for $n \geq 2$, and since $p^\star(1) = 2^{1-1}$, we have $p^\star(n) = 2^{n-1}$.

Generating Functions. Several counting problems are easily resolved by evaluating or simplifying their generating function. This is particularly true of functions that are linear combinations of geometric sequences.

Since $p_k^\star(n)$ counts the number of k-tuples $(a_1, \ldots, a_k)$ of positive integers whose sum is n, it equals the coefficient of x^n in the expansion $(x+x^2+x^3+\cdots)^k =$

$x^k(1-x)^{-k}$. Hence the sequence $\{p^\star(n)\}_{n\geq 1}$ has the generating function

$$\sum_{n=1}^{\infty} p^\star(n)x^n = \sum_{k=1}^{\infty} x^k(1-x)^{-k} = \frac{x}{1-2x} = x\sum_{n=0}^{\infty}(2x)^n.$$

Comparing coefficients of x^n, we have $p^\star(n) = 2^{n-1}$.

Sets. We know that if S has n elements, then the power set $\mathcal{P}(S)$ of S has 2^n elements. Therefore it must be the case that the power set of a set of size $n-1$ must be in one-to-one correspondence with the set of compositions of n, given the formula in (2). We exploit this fact to produce such a one-to-one correspondence between the two sets.

Write $\mathcal{P}^\star(n)$ for the set of all ordered tuples $(a_1,\ldots,a_k)$ of positive integers whose sum is n, $\mathcal{P}(S)$ for the power set of S, and $[m]$ for $\{1,\ldots,m\}$ where $m\in\mathbb{N}$. Define a mapping $\varphi:\mathcal{P}^\star(n)\to\mathcal{P}\Big([n-1]\Big)$ by

$$(a_1,\ldots,a_k)\mapsto\{a_1, a_1+a_2,\ldots,a_1+a_2+\cdots+a_{k-1}\},$$

with $(n)\mapsto\emptyset$.

We show that φ is *one-one* and *onto*. Suppose $\mathbf{a} := (a_1,\ldots,a_r)$ and $\mathbf{b} := (b_1,\ldots,b_s)$ are unequal elements in $\mathcal{P}^\star(n)$. So $a_i = b_i$ for $1\leq i<j$, and $a_j<b_j$ for a suitable j. Then $a_1+\cdots+a_j\in\varphi(\mathbf{a})\setminus\varphi(\mathbf{b})$ since

$$b_1+\cdots+b_{j-1} = a_1+\cdots+a_{j-1} < a_1+\cdots+a_{j-1}+a_j < b_1+\cdots+b_{j-1}+b_j,$$

and the elements of $\varphi(\mathbf{b})$ are arranged in increasing order. Hence φ is one-one.

Any non-empty subset $\{b_1,\ldots,b_k\}$ of $[n-1]$, with $b_1<\cdots<b_k$, is the image of $(b_1, b_2-b_1, b_3-b_2,\ldots,b_k-b_{k-1}, n-b_k)\in\mathcal{P}^\star(n)$, as can be easily verified. Hence φ is also onto.

Functions. It is customary for B^A to denote the set of all functions with domain A and codomain B, since $\left|B^A\right| = |B|^{|A|}$ for finite sets A, B. So if we denote by $\mathbf{2}$ the set $\{0,1\}$, by $\mathbf{2}^S$ we mean the set of all functions from S into $\mathbf{2}$, and this set has size $|\mathbf{2}|^{|S|} = 2^{|S|}$. Borrowing the notations from Sets, we must now exhibit a one-to-one correspondence between the sets $\mathcal{P}^\star(n)$ and $\mathbf{2}^{[n-1]}$. Recall the characteristic function of a set A is a function χ_A defined as

$$\chi_A(a) = \begin{cases} 1 & \text{if } a\in A\\ 0 & \text{if } a\notin A\end{cases}.$$

Define a mapping $\psi:\mathcal{P}^\star(n)\to\mathbf{2}^{[n-1]}$ by

$$(n)\mapsto\chi_\emptyset = \mathbf{0},\quad (a_1,\ldots,a_k)\mapsto\chi_A,$$

where $A = \{a_1, a_1+a_2,\ldots,a_1+a_2+\cdots+a_{k-1}\}$.

We show that ψ is *one-one* and *onto*. With the notations used for Sets, recall that $a_1 + \cdots + a_j \in \varphi(\mathbf{a}) \setminus \varphi(\mathbf{b})$. So if $A = \{a_1, a_1 + a_2, \ldots, a_1 + a_2 + \cdots + a_{r-1}\}$ and $B = \{b_1, b_1 + b_2, \ldots, b_1 + b_2 + \cdots + b_{s-1}\}$, then $\chi_A \neq \chi_B$. Hence ψ is one-one.

Suppose $f \in \mathbf{2}^{[n-1]}$, with $f \neq \mathbf{0}$. Let $f^{-1}(1) = \{a_1, \ldots, a_k\}$, where $a_1 < \ldots < a_k$. Then it is easy to verify that f is the image under ψ of $(a_1, a_2 - a_1, a_3 - a_2, \ldots, a_k - a_{k-1}, n - a_k) \in \mathcal{P}^\star(n)$. Hence ψ is also onto.

Inclusion & Exclusion. The Principle of Inclusion & Exclusion (PIE) is useful in solving such diverse problems as counting the number of derangements, the number of onto mappings between two finite sets, and number of integers relatively prime to and less than a given integer. PIE counts the number of elements outside of finitely many (finite) sets, giving the result in terms of intersections of these sets. When only two sets are involved, PIE is just the formula

$$|A \cup B| + |A \cap B| = |A| + |B|,$$

for *finite* sets A, B. We include this connection as our last example, but this connection also relies on induction.

We also induct on n. Observe that $p^\star(1) = 1$, and assume that $p^\star(m) = 2^{m-1}$ for $1 \leq m \leq n$. To each $\mathbf{a} = (a_1, \ldots, a_k) \in \mathcal{P}^\star(n)$, we associate two elements of $\mathcal{P}^\star(n+1)$, given by

$$\ell(\mathbf{a}) := (1, a_1, \ldots, a_k), \qquad r(\mathbf{a}) := (a_1, \ldots, a_k, 1).$$

Define

$$\begin{aligned} \mathcal{L}^\star(n) &:= \ell\Big(\mathcal{P}^\star(n)\Big) = \{\ell(\mathbf{a}) : \mathbf{a} \in \mathcal{P}^\star(n)\}, \\ \mathcal{R}^\star(n) &:= r\Big(\mathcal{P}^\star(n)\Big) = \{r(\mathbf{a}) : \mathbf{a} \in \mathcal{P}^\star(n)\}; \\ \mathcal{Q}^\star(n) &:= \mathcal{P}^\star(n) \setminus \Big\{\mathcal{L}^\star(n-1) \cup \mathcal{R}^\star(n-1)\Big\}. \end{aligned}$$

Therefore

$$|\mathcal{P}^\star(n)| = |\mathcal{L}^\star(n-1) \cup \mathcal{R}^\star(n-1)| + |\mathcal{Q}^\star(n)| \tag{3}$$

Clearly $|\mathcal{L}^\star(n)| = |\mathcal{P}^\star(n)| = |\mathcal{R}^\star(n)|$, and these equal 2^{n-1} by induction hypothesis. We claim that $|\mathcal{L}^\star(n-1) \cap \mathcal{R}^\star(n-1)| = 2^{n-2}$. Any $\mathbf{a} \in \mathcal{L}^\star(n-1) \cap \mathcal{R}^\star(n-1)$ must have first and last part 1. Removing these 1's results in an element in $\mathcal{P}^\star(n-1)$. Conversely, to any element in $\mathcal{P}^\star(n-1)$ we can attach a 1 at both ends and obtain an element in $\mathcal{L}^\star(n-1) \cap \mathcal{R}^\star(n-1)$. Hence $\mathcal{L}^\star(n-1) \cap \mathcal{R}^\star(n-1)$ is in one-to-one correspondence with $\mathcal{P}^\star(n-1)$, and so by induction hypothesis $|\mathcal{L}^\star(n) \cap \mathcal{R}^\star(n)| = 2^{n-2}$. Therefore

$$\begin{aligned} |\mathcal{L}^\star(n) \cup \mathcal{R}^\star(n)| &= |\mathcal{L}^\star(n)| + |\mathcal{R}^\star(n)| - |\mathcal{L}^\star(n) \cap \mathcal{R}^\star(n)| \\ &= 2^{n-1} + 2^{n-1} - 2^{n-2} = 3 \cdot 2^{n-2} \end{aligned} \tag{4}$$

Note that $\mathbf{a} = (a_1, \ldots, a_k) \in \mathcal{Q}^\star(n) \subset \mathcal{P}^\star(n)$ if and only if $a_1 > 1$ and $a_k > 1$. The correspondence $(a_1, \ldots, a_k) \leftrightarrow (a_1 - 1, a_2, \ldots, a_{k-1}, a_k - 1)$ sets up a one-to-one correspondence between $\mathcal{Q}^\star(n)$ and $\mathcal{P}^\star(n-2)$. Therefore

$$|\mathcal{Q}^\star(n)| = 2^{n-3} \tag{5}$$

From equations (3), (4), (5) we get

$$p^\star(n+1) = |\mathcal{P}^\star(n+1)| = |\mathcal{L}^\star(n) \cup \mathcal{R}^\star(n)| + |\mathcal{Q}^\star(n+1)| = (3 \cdot 2^{n-2}) + 2^{n-2} = 2^n.$$

References

[1] Peter J. Cameron, *Combinatorics: Topics, Techniques, Algorithms*, Cambridge University Press, 1994.

[2] Louis Comtet, *Advanced Combinatorics*, Springer, 2001.

[3] Ross Honsberger, *Mathematical Morsels*, The Dolciani Mathematical Expositions, Mathematical Association of America, 1980.

[4] Marko Petkovsek, Herbert Wilf and Doron Zeilberger, *$A = B$*, A. K. Peters, 1997.

[5] Herbert J. Ryser, *Combinatorial Mathematics*, The Carus Mathematical Monographs, # 14, Mathematical Association of America, 1963.

[6] Fred S. Roberts, *Applied Combinatorics*, Prentice-Hall, 1984.

[7] J. H. van Lint and R. M. Wilson, *A Course in Combinatorics*, Cambridge University Press, 1993.

[8] Herbert S. Wilf, *generatingfunctionology*, Academic Press, 1993.

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