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L3

LVP 1.5-1.8
HHM 2.1, 2.2

Fundamental Theorem of Arithmetic

Every positive integer n decomposes uniquely into a product of prime numbers.

Ex. $15 = 3^1 \cdot 5^1$
$240 = 2^4 \cdot 3 \cdot 5$

Q. How many factors does 240 have?
A. Since primes do not have non-trivial factors, all the factors of 240 must themselves be products of the prime factors of 240. Some examples:

$$2,\ 2\cdot 3,\ 2\cdot 3\cdot 5,\ 2^2\cdot 3\cdot 5$$

How can we count these. Easy: think of multisets!

$240 \longleftrightarrow \{2,2,2,2,3,5\}$ ← multiset

$2 \longleftrightarrow \{2\}$
$2\cdot 3 \longleftrightarrow \{2,3\}$
$2\cdot 3\cdot 5 \longleftrightarrow \{2,3,5\}$
$2^2\cdot 3\cdot 5 \longleftrightarrow \{2,2,3,5\}$

sub-multisets.

Recall our encoding strategy for counting multisets:

$$\{2,2,3,5\} = \frac{2}{2}\ \frac{1}{3}\ \frac{1}{5}$$

$\therefore$ # sub-multisets $= 5 \cdot 2 \cdot 2 = 20$

ie. 240 has 20 distinct factors.

Proposition The set $\{1, 2, \ldots, n\}$ has the same # subsets of even size as subsets of odd size.

Pf Trick is to show that there is a bijection between the two subsets. That way, we don't need to count the subsets at all!

Consider an element of $\{1, 2, \ldots n\}$, say 1. It is either an element of a given subset or it isn't. If 1 is present, remove it to generate a new subset. If 1 is absent, add it to generate a new subset. This is our function. It is bijective:

$$\{\cdot\cdot\cdot|\cdot\cdot\} \underset{f}{\overset{f}{\rightleftarrows}} \{\circ\circ\circ\circ\circ\}$$

Let the domain of f be the even-sized subsets. Then the image is the odd-sized subsets. Since the domain and image are of equal size for a bijection, we have proved that # even subsets = # odd subsets.

<u>EX</u> $n=3$:

$$\begin{array}{ccc}
\phi & \longleftrightarrow & \{1\}. \\
\{1,2\} & \longleftrightarrow & \{2\}. \\
\{1,3\} & \longleftrightarrow & \{3\}. \\
\{2,3\} & \longleftrightarrow & \{1,2,3\}.
\end{array}$$

Q. How many words of 4 letters have no repeats?

A. $\underset{26}{\_}\ \underset{25}{\_}\ \underset{24}{\_}\ \underset{23}{\_} = 26 \cdot 25 \cdot 24 \cdot 23$

$$= \frac{26!}{(26-4)!}$$

In general, the # words of length k from an alphabet of size n with no repeated letters is:

$$n(n-1)\cdots(n-(k-1)) = \frac{n!}{(n-k)!}$$

Q. How many ways are there to permute (line up) n (distinguishable/labelled) objects?

A $n(n-1)\cdots 2\cdot 1 = n!$
↑
ways to pick 1st object

Q. How many words of length 4 with no repeated letters are in alphabetical order?

A First pick a word, eg ABDC. There are $26!/(26-4)!$ ways to do this.

Consider all words comprised of, say, $\{A, B, C, D\}$. There are $4!$ of them, but only one of them has its letters ordered alphabetically.

Thus we can arrange all $26!/(26-4)!$ words in groups of size $4!$, where each group comprises all possible orderings of a set of 4 letters. The number of such groups is $[26!/(26-4)!]/4!$.

Q. How many ways are there to flip a coin 4 times and get exactly 2 heads?

A Emmunerate:

HHTT   THHT
HTHT   THTH
HTTH   TTHH.

Seems we had to choose two out of four positions to place the 2 heads. Think of the positions as the alphabet in the previous problem:

$$A = \{1, 2, 3, 4\}$$

We want to create "words" of length 2 (2 heads) from A with no repeated letters:

12   23
13   24
14   34

Not quite. There are other words that fit the bill:

21   32
31   42
41   43

Mapping back to our problem

$$12 = \underset{1\ 2}{HH}TT = \underset{2\ 1}{HH}TT = 21$$

Thus we only want to count # words in "alphabetical" order. The previous problem told us that the answer is:

$$\frac{4!}{(4-2)!\,2!}$$

In general, the number of k-element subsets of an n-element set is

$$\binom{n}{k} = \frac{n!}{(n-k)!\,k!}$$

This is usually called "n choose k".

There are 2^n ways to flip a coin n times. For each sequence we can count the # heads, k. Put all sequences w/ same # heads into their own set. There are $(n+1)$ such sets $(k = 0, 1, 2, \ldots, n)$. They are disjoint: a sequence cannot have both 5 and 2 heads, say.

Thus:

$$\{\text{all sequences}\} = \{\text{seq w/ 0 heads}\} \cup \{\text{seq w/ 1 head}\} \cup \vdots \; \{\text{seq w/ n heads}\}. \quad (*)$$

Now:

$$\#\{\text{all seqs}\} = 2^n$$

$$\#\{\text{seq w/ k heads}\} = \binom{n}{k}$$

Thus $(*) \Rightarrow$

$$2^n = \binom{n}{0} + \binom{n}{1} + \cdots \binom{n}{n} = \sum_{k=0}^{n} \binom{n}{k}$$

This is an example of the binomial th$^{\underline{m}}$ (binomial because there are 2 options on each flip) and explains why $\binom{n}{k}$ is often called a binomial coefficient.

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Note: $\binom{n}{0} = \frac{n!}{(n-0)!\,0!}$

Now:
$$0! = 1$$
since there is only one way to permute an empty set of objects.

Consider the set $\{1, 2, \ldots, n\}$. We know that # k-element subsets is $\binom{n}{k}$. Consider the subsets w/ an even # elements. They number
$$\binom{n}{0} + \binom{n}{2} + \cdots + \binom{n}{2\lfloor \frac{n}{2} \rfloor}$$
$\lfloor \frac{x}{y} \rfloor$ = largest integer $\leq \frac{x}{y}$

in total. But we proved at the beginning of the lecture that this number equals:
$$\binom{n}{1} + \binom{n}{3} + \cdots + \binom{n}{2\lfloor \frac{n}{2} \rfloor + 1}$$

which must therefore be
$$\frac{2^n}{2} = 2^{n-1}.$$

This allows us to write down the non-obvious formula:

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} = 0$$

Cool!