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LVP 1.5-1.8

L3

HAM 2.1, 2.2

Fundamental Theorem of Arithmetic

Every positive integer n decomposes uniquely into a product of prime numbers.

EX. $15 = 3^1 \cdot 5^1$
 $240 = 2^4 \cdot 3 \cdot 5$

Q How many factors does 240 have?

A. Since primes do not have non-trivial factors, all the factors of 240 must themselves be products of the prime factors of 240. Some examples:

$2, 2 \cdot 3, 2 \cdot 3 \cdot 5, 2^2 \cdot 3 \cdot 5$

How can we count these. Easy: Think of multisets!

$240 \leftrightarrow$	$\{2, 2, 2, 2, 3, 5\}$	\leftarrow multiset
$2 \leftrightarrow$	$\{2\}$	} sub-multisets.
$2 \cdot 3 \leftrightarrow$	$\{2, 3\}$	
$2 \cdot 3 \cdot 5 \leftrightarrow$	$\{2, 3, 5\}$	
$2^2 \cdot 3 \cdot 5 \leftrightarrow$	$\{2, 2, 3, 5\}$	

Recall our encoding strategy for counting multisets:

$$\{2, 2, 3, 5\} = \frac{2}{2} \frac{1}{3} \frac{1}{5}$$

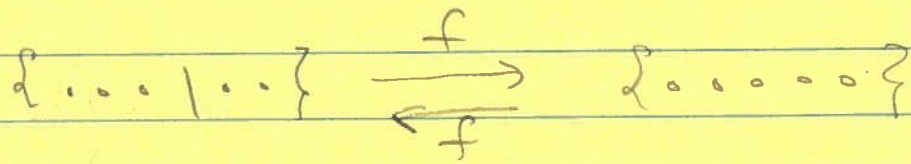
$$\therefore \# \text{ sub-multisets} = 5 \cdot 2 \cdot 2 = 20$$

ie. 240 has 20 distinct factors.

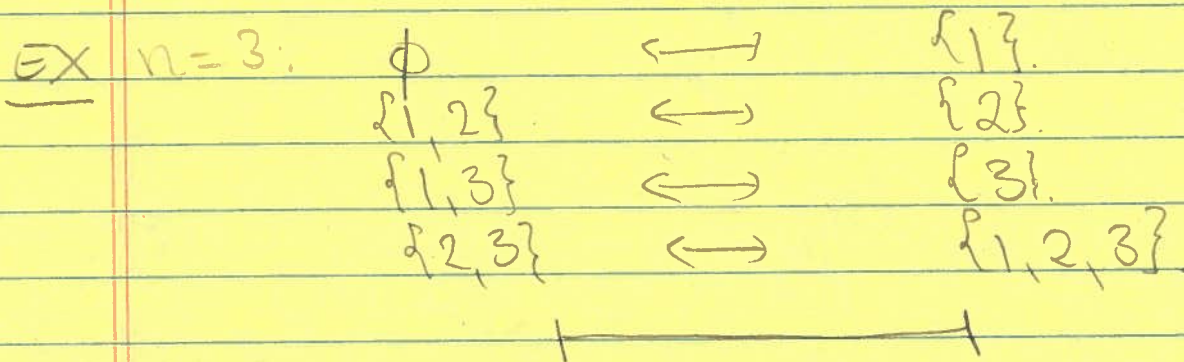
Proposition The set $\{1, 2, \dots, n\}$ has the same # subsets of even size as subsets of odd size.

Pr Trick is to show that there is a bijection between the two subsets. That way, we don't need to count the subsets at all!

Consider an element of $\{1, 2, \dots, n\}$, say 1. It is either an element of a given subset or it isn't. If 1 is present, remove it to generate a new subset. If 1 is absent, add it to generate a new subset. This is our function. It is bijective.



Let the domain of f be the even-sized subsets. Then the image is the odd-sized subsets. Since the domain and image are of equal size for a bijection, we have proved that $\# \text{ even subsets} = \# \text{ odd subsets}$.



Q. How many words of 4 letters have no repeats?

A

$$\begin{aligned} & \underline{26} \quad \underline{25} \quad \underline{24} \quad \underline{23} & = & 26 \cdot 25 \cdot 24 \cdot 23 \\ & & = & \frac{26!}{(26-4)!} \end{aligned}$$

In general, the # words of length k from an alphabet of size n with no repeated letters is:

$$n(n-1) \cdots (n-(k-1)) = \frac{n!}{(n-k)!}$$

Q. How many ways are there to permute (line up) n (distinguishable/labelled) objects?

A $n(n-1) \cdots 2 \cdot 1 = n!$

↑
#ways
to pick
1st object

Q. How many words of length 4 with no repeated letters are in alphabetical order?

A First pick a word, eg ABDC. There are $26!/(26-4)!$ ways to do this.

Consider all words comprised of, say, $\{A, B, C, D\}$. There are $4!$ of them, but only one of them has its letters ordered alphabetically.

Thus we can arrange all $26!/(26-4)!$ words in groups of size $4!$, where each group comprises all possible orderings of a set of 4 letters. The number of such groups is $[26!/(26-4)!] / 4!$.

Q How many ways are there to flip a coin 4 times and get exactly 2 heads?

A Enumerate:

HHTT T#HT
HTHT THTH
HTTH TT#H.

Seems we had to choose two out of four positions to place the 2 heads. Think of the positions as the alphabet in the previous problem:

$$A = \{1, 2, 3, 4\}$$

We want to create "words" of length 2 (2 heads) from A with no repeated letters:

12 23
13 24
14 34

Not quite. There are other words that fit the bill:

21 32
31 42
41 43

Mapping back to our problem

$$12 = \underset{1\ 2}{\text{HHTT}} = \underset{2\ 1}{\text{HHTT}} = 21$$

Thus we only want to count # words in "alphabetical" order. The previous problem told us that the answer is:

$$\frac{4!}{(4-2)! \cdot 2!}$$



In general, the number of k -element subsets of an n -element set is

$$\binom{n}{k} = \frac{n!}{(n-k)! \cdot k!}$$

This is usually called " n choose k ".

There are 2^n ways to flip a coin n times. For each sequence we can count the # heads, k . Put all sequences w/ same # heads into their own set. There are $(n+1)$ such sets ($k=0, 1, 2, \dots, n$). They are disjoint: a sequence cannot have both 5 and 2 heads, say.

Thus:

$$\begin{aligned} \{\text{all sequences}\} &= \{\text{seq w/ 0 heads}\} \cup \\ &\quad \{\text{seq w/ 1 head}\} \cup \\ &\quad \vdots \\ &\quad \{\text{seq w/ n heads}\}. \quad (*) \end{aligned}$$

Now:

$$\# \{\text{all seqs}\} = 2^n$$

$$\# \{\text{seq w/ k heads}\} = \binom{n}{k}$$

Thus (*) \Rightarrow

$$2^n = \binom{n}{0} + \binom{n}{1} + \dots + \binom{n}{n} = \sum_{k=0}^n \binom{n}{k}$$

This is an example of the binomial theorem (binomial because there are 2 options on each flip) and explains why (*) is often called a binomial coefficient.

Binomial
Theorem

$$(x+y)^n = \sum_{k=0}^n \binom{n}{k} x^k y^{n-k}$$

Note:
$$\binom{n}{0} = \frac{n!}{(n-0)! \cdot 0!}$$

Now:
$$0! = 1$$

since there is only one way to permute an empty set of objects.



Consider the set $\{1, 2, \dots, n\}$. We know that # k -element subsets is $\binom{n}{k}$. Consider the subsets w/ an even # elements. Their number

$$\binom{n}{0} + \binom{n}{2} + \dots + \binom{n}{2 \lfloor \frac{n}{2} \rfloor}$$

$\lfloor \frac{x}{y} \rfloor = \text{largest integer } \leq \frac{x}{y}$

in total. But we proved at the beginning of the lecture that this number equals;

$$\binom{n}{1} + \binom{n}{3} + \dots + \binom{n}{2 \lfloor \frac{n}{2} \rfloor + 1}$$

which must therefore be

$$\frac{2^n}{2} = 2^{n-1}$$

This allows us to write down the non-obvious formula:

$$\sum_{k=0}^n (-1)^k \binom{n}{k} = 0.$$

Cool!