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L2 <u>Starting to Count</u>

Reading: §1.1-1.3 LVP, §2.1 HHM

Q How many "words" have four letters? Assume the alphabet has 26 letters.

A. Here is a "word" w/ four letters: AABB
In general a word looks like _ _ _ _
where there are 26 choices for the first letter slot, 26 for the second, etc.

Thus there are 26^4 possible 4-letter words.

The fact that in an alphabet of m "letters", there are m^n "words" of length n is known as the <u>Multiplication Principle</u>. We already used it in the previous lecture: $\exists$ 2^n sequences of n coin flips.

Q. How many functions are there from $\{1, 2, \ldots, n\}$ to $\{1, 2, \ldots, m\}$?

A. A function is specified by its output on each element of its domain.

$f(1) =$ _ (m choices)
$f(2) =$ _ (m choices)
$\vdots$
$f(n) =$ _ (m choices)

This gives a "word",

$$\underset{1}{\_}\ \underset{2}{\_}\ \cdots\ \underset{n}{\_},$$

of length n with an alphabet $\{1, 2, \ldots, m\}$. Thus there are m^n functions.

Q. How many subsets are there of $\{1, 2, \ldots, n\}$?

A. Let's build a subset $S \subseteq \{1, 2, \ldots, n\}$

Is 1 in it? _ (yes/no)

Is 2 in it? __

⋮

Is n in it? __

Thus each subset corresponds to a word,

$$\underset{1}{\_}\ \underset{2}{\_}\ \cdots\ \underset{n}{\_},$$

constructed from an alphabet of 2 letters. So, there are 2^n subsets of $\{1, 2, \ldots n\}$.

One thing that is common to all these examples is that we are making independent choices in each "slot". In this sense, these counting problems are very similar to coin tossing:

Consider a "word" of length n taken from the alphabet $\{H, T\}$.

The set of positions of all H's gives a subset of $\{1, 2, \ldots, n\}$, eg $HHT \longrightarrow \{1,2\} \subset \{1,2,3\}$.

Given a subset, we can write down the unique word corresponding to it. The algorithm is:

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for i = 1, ..., n:
    if i in S:
        put H in slot i
    else:
        put T in slot i
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eg $\{1,4\} \subset \{1,2,3,4\} \longrightarrow HTTH$.

There is a one-to-one correspondence between words and subsets. In other words, the mapping between the set of words, eg $A = \{HHH, HHT, HTH, \ldots\}$, and the set of subsets, $B = \{\{1,2,3\}, \{1,2\}, \{1,3\}, \ldots\}$ is a BIJECTION. Note A and B must have the same size $|A| = |B|$.

Def^{n} A multiset is like a set, but can have repeated elements.

Q $\{1,1,2\}$ is a multiset but not a set. How many submultisets does it have?

A Let's enumerate:

$\phi, \{1\}, \{2\}, \{1,1\}, \{1,2\}, \{1,1,2\}$

TRICK: Let us now develop a mapping between these multisets and words.

Let's choose the (common) length of the words to be the number of unique numbers in the original multiset, which is 2 in this case:

	?	?
multiset element:	1	2

To determine what "letter" goes in each slot, count the number of occurrences of the corresponding multiset element in the given submultiset:

$\phi = \underset{1}{\underline{0}} \; \underset{2}{\underline{0}}$

$\{1\} = \underset{1}{\underline{1}} \; \underset{2}{\underline{0}}$

$\{2\} = \underset{1}{\underline{0}} \; \underset{2}{\underline{1}}$

$\{1,1\} = \underset{1}{\underline{2}} \; \underset{2}{\underline{0}}$

$\{1,2\} = \underset{1}{\underline{1}} \; \underset{2}{\underline{1}}$

$\{1,1,2\} = \underset{1}{\underline{2}} \; \underset{2}{\underline{1}}$

Notice that again the mapping is one-to-one: once we know the #1's and #2's in a multiset, we know the multiset because order doesn't matter!

What is the alphabet in this case? Well, things are little more complicated than before because there are three "letters" we can place in slot 1 (0, 1, 2) but only 2 "letters" may be placed in slot 2 (0, 1). That is, the alphabet is slot-dependent.

Nevertheless, this mapping helps us to see how to count sub-multisets: multiply the sizes of the alphabets:

$$|\{0,1,2\}| \cdot |\{0,1\}|$$

$$= 3 \cdot 2 = 6.$$

We didn't need this trick for such a simple problem, but it becomes indispensable for the following problem

Q How many multisets are there of $S = \{1,1,2,2,\ldots,n\}$?

A. Using our trick, words are of length n (the # unique elements of S and look like:

$$\text{submultiset} = \underline{\#1}\ \underline{\#2}\ \ldots\ \underline{\#n}$$

Now $\#1 \in \{0,1,2\}$

$\#2 \in$ "

$\vdots$

$\#n \in \{0,1,2\}$.

Thus # multisets $= \underbrace{3 \cdot 3 \cdots 3}_{n \text{ times}} = 3^n$

Q How many sub-multisets for

$$\{\underbrace{1, \ldots, 1}_{m}, \underbrace{2, \ldots 2}_{m}, \ldots \underbrace{n, \ldots n}_{m}\}.$$

A There are n slots in each word, as before, but now $(m+1)$ choices for each slot (instead of 3). Thus the answer is $(m+1)^n$.

Special case: number of subsets of $\{1, 2, \ldots, n\}$ is $(\underset{\uparrow m}{1}+1)^n = 2^n$, cf p10 of LVP.

The trick of representing things as "words" is often call "encoding" in math.