L16

Pigeonhole Principle

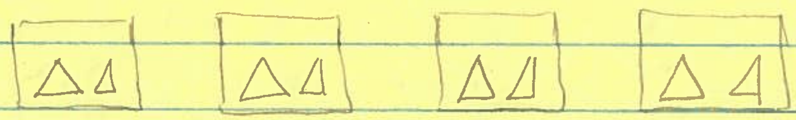
Q. How many people do you need in a room to be guaranteed that there is a shared birthday among them?

324 LPV
324 HHM

A. 365 possible birthdays $\Rightarrow$ 366 or more people

Pigeonhole Principle

If you put n pigeons into $m\ (<n)$ pigeonholes then at least one hole receives at least n/m pigeons:



6 "pigeons"

4 pigeonholes

$\Rightarrow$ most "uniform" distribution is $\frac{6}{4}$ or $1\frac{1}{2}$ "pigeons" per hole.

If we don't allow fractional pigeons, then at least one hole contains at least $\lceil n/m \rceil$ pigeons:

$\lceil 1\frac{1}{2} \rceil = 2$

Utility of Pigeon hole Principle

case $n<m$

$$\frac{n}{m} < 1 \Rightarrow \left\lceil \frac{n}{m} \right\rceil = 1$$

ie. at least one hole contains at least one pigeon.

This fact is rarely useful in problem solving.

case $n>m$

$$\frac{n}{m} > 1 \Rightarrow \left\lceil \frac{n}{m} \right\rceil \geq 2$$

ie. at least one hole contains at least two pigeons.

This fact is often useful in problem solving.

Though simple to state (and prove, p 3), the Pigeonhole Principle is powerful: it can be used to prove very non-obvious results!

Formal proof of Pigeonhole Principle.

Let X_i be the number of pigeons in hole i.

Argue by contradiction. Suppose $X_i < n/m$, $i=1,\dots,m$; then $X_1 + \cdots + X_m < \frac{n}{m} \cdot m = n$. But $X_1 + \cdots + X_m = n$. Thus, at least one X_i must be greater than or equal to n/m ☒

Q. Deal out ten 5-card poker hands from a standard deck. Show that at least one person has two spades.

A. The people are the "holes", and the spades are the "pigeons". We know there are 10 holes, but how many pigeons are there?

A standard deck contains 13 spades. How many of these were given to the players? We don't know exactly, but we can provide a lower bound, as follows:

cards dealt = $10 \cdot 5 = 50$.

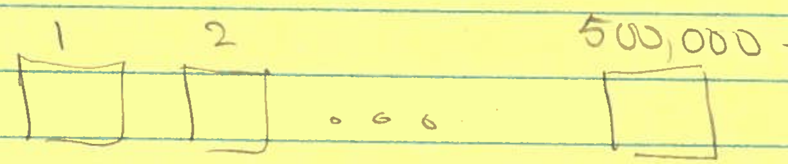
\# cards in deck = 52
$\Rightarrow$ 2 cards not dealt.
$\Rightarrow$ $11 \leq$ #spades dealt ≤ 13
$\downarrow$
worst-case scenario:
2 spades not dealt.

If 11 spades were dealt to 10 players, then at least one received at least $\lceil 11/10 \rceil = 2$ spades.

[If 13 spades were dealt, then at least one received at least $\lceil 13/10 \rceil = 2$ spades.]

Q. Are there any people in NYC who share the same # hairs on their head? Assume no one has more than 500,000 hairs and that there are 8,000,000 people in NYC.

A. Create a box for each possible hair number per head, ie.



Ask each person to place their name (pigeon) in the box labelled by the # hairs on their head.

Then at least one box will have at least $\lceil 8{,}000{,}000 / 500{,}000 \rceil = 16$ names, i.e. at least 16 people share the same # hairs!

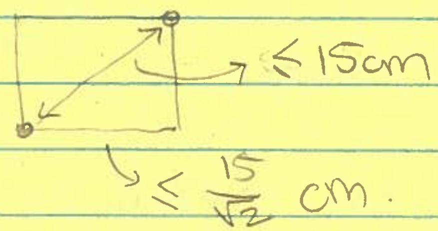
Q. We shoot 50 times at a square target with side length 70 cm. Prove that two shots land closer than 15 cm.

15 cm.

A. The pigeons are the shots. But what are the pigeonholes? Sub-regions of the target! There are a few constraints:

1. their shape must match the target (square) s.t. every shot falls into some pigeonhole.

2. Squares should be small enough that no two points within a square are more than 15 cm apart:



Then, all we have to demonstrate is that at least one square contains 2 shots.

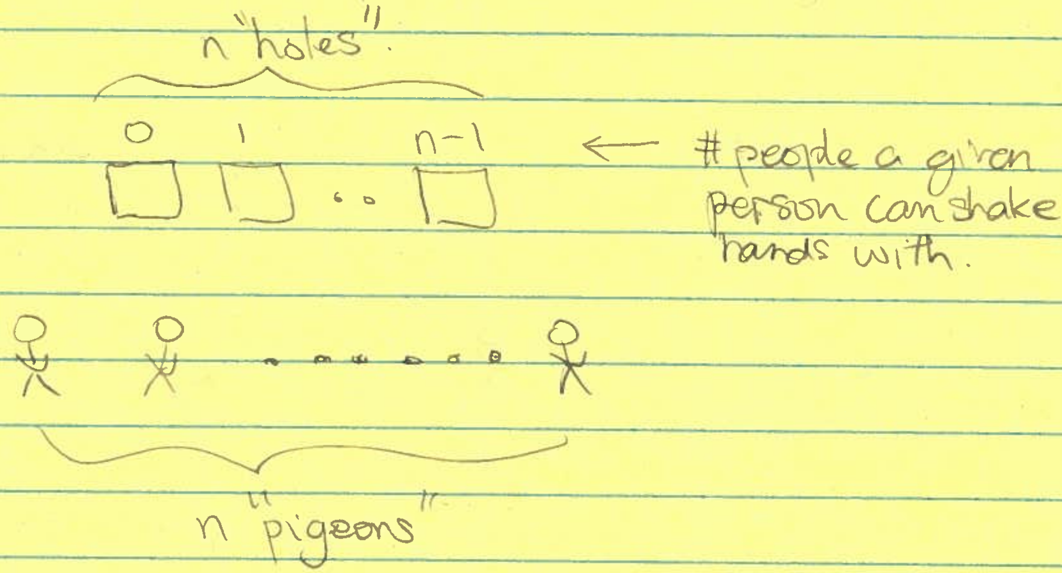
Let's choose squares of size $10\text{ cm} \leq \frac{15}{\sqrt{2}}\text{ cm}$. Then we can fit 49 ($=7\times7$) of them into the target. Thus we must distribute 50 shots among 49 squares. By the pigeonhole principle, at least one square contains two shots.

What if we had chosen smaller squares, eg $5\text{cm} \times 5\text{cm}$? Then we would have to distribute 50 shots among $14\times14 = 196$ squares. Since $196 > 50$, the pigeonhole principle doesn't help us.

Thus we see that the trick in this solution is to choose the square size *small enough* that shots within a square are guaranteed to be within 15cm of each other, while *large enough* that they are out-numbered by shots!

Ex Prove that, in a group of people (more than 1 person) there is always a pair of people who shake hands with the same number of people.

Sol^n



Seems like we cannot usefully apply the Pigeonhole Principle since pigeons don't outnumber holes.

But there is something special about first and last holes: they can't both be occupied! For if a person shakes hands with no-one (pigeon in 1st hole) then everybody else shakes hands with at most $n-2$ people, ie. there is no pigeon in last hole. Similarly, if there is a pigeon in last hole, then someone shook hands with everybody else, so it is impossible that someone shook hands with no-one. Thus the # of holes that can be pseudo-independently* occupied is n-1 (eliminate hole 0 or hole n-1). Thus # holes < # pigeons. Thus, by Pigeonhole Principle, there is one hole occupied by at least two pigeons, ie. at least two people shook hands w/ the same # people.

*: you still have the constraint that sum of people in (n-1) holes is n.

Q. Prove that any 10 distinct numbers between 1 and 100 contains two disjoint non-empty subsets with the same sum, e.g.

$\{1, 2, \ldots, 9, 10\}$

$\{1, 10\}$ $\{2, 9\}$

$1 + 10 = 11 = 2 + 9$

A. What are the pigeons? The subsets.
What are the pigeonholes? There is one for each possible value of the sum of elements in a subset.

Thus we need to count subsets and distinct sums:

pigeons: $\#\text{subsets} = \underbrace{2^{10} - 1}_{\text{omit empty subset}} = 1023$

holes: $\#\text{sums} = (\text{max sum}) - (\text{min sum}) + 1$

$= \underbrace{(100 + 99 + \cdots + 91)}_{\text{10-element subset w/ largest numbers}} - \underbrace{(1)}_{\text{1-element subset w/ smallest number}} + 1$

At this point, we just need to convince ourselves that there are more pigeons than holes. Thus an upper bound on the

holes suffices, if its smaller than the # pigeons.

$[\#\text{holes} < 100 \cdot 10 \cdot = 1000] < [\#\text{pigeons} = 1023]$

Thus pigeonhole principle says there is at least one sum that corresponds to two subsets.

But these two subsets may share elements. No problem! If that occurs, then remove those common elements to form 2 disjoint subsets. The new sums will still be equal because the same numbers have been subtracted from both original sums!

Q. Same Q but w/ 100 replaced by k. What is the largest value of k for which we may prove the statement using the Pigeonhole Principle.

A. The only change is to the # sums (holes), which becomes

$$k + (k-1) + \cdots + (k-9)$$

$$= 10k - [1 + 2 + \cdots + 9]$$

$$= 10k - \frac{9(10)}{2} = 10k - 45$$

For pigeonhole principle to "work,"

holes < # pigeons.

$\Rightarrow \quad 10k - 45 < 1023$

$\Rightarrow \quad 10k < 1068$

$\Rightarrow \quad k < 106.8$, i.e. $k \leq 106$.

Q Prove that any 6-subset of $\{1, 2, \ldots, 9\}$ must contain two elements that sum to 10.

A. What are the possible pairs of elements That sum to 10?

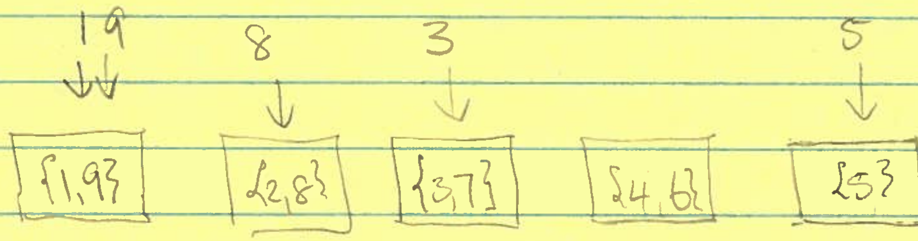
$\{1, 9\}, \{2, 8\}, \{3, 7\}, \{4, 6\}$

Let those, together with $\{5\}$, be the pigeon holes. (5 holes).

Let the pigeons be the elements of a 6-subset e.g.

$\{1, 3, 5, 7, 8, 9\}$ (6 pigeons)

Then place pigeons in holes as follows:



Since # pigeons > # holes, Pigeonhole Principle implies that 2 elements of a 6-subset will fall into one of the holes. That hole can't be the one labelled $\{5\}$, since there is only element (5) that can drop into that hole. Thus, one of the other 4 holes must contain two elements of the subset. Those two elements must be the labels on the hole. Therefore those two elements sum to 10.