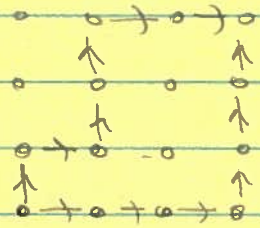
L14 **Lattice Paths, Catalan Numbers, and André's Reflection Principle**

§2.6.6 HMM Suppose you live on a grid at (0,0) and you work at (3,3). You can only walk along grid edges.

Q What is the fewest number of steps to get to work?

A 6. No matter your path, you must traverse 3 horizontal edges, and 3 vertical edges.



Such paths are called "minimal length" paths

Q How many minimal length paths are there?

A Viewing each path as a word of length 6 containing exactly 3 vertical steps

eg

$\uparrow \rightarrow \uparrow \uparrow \rightarrow \rightarrow$

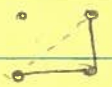
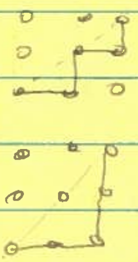
$\rightarrow \rightarrow \rightarrow \uparrow \uparrow \uparrow$

We see that there must be $\binom{6}{3}$ minimal length paths.

Q. How many minimal length paths from $(0,0)$ to (m,n)?

A. $\binom{m+n}{m} = \binom{m+n}{n}$ (We saw this in L9)

Now suppose $m=n$. How many of the $\binom{2n}{n}$ minimal-length paths (sometimes called "walks") are confined to the triangle below the main diagonal, $x=y$?

n=1	n=2	n=3
		 EEENNN EENENN EENNEN ENEENN ENENEN

# favorable paths	1	2	5
# possible paths $\binom{2n}{n}$	2	6	20.
n+1	2	3	4

Conjecture: # favorable paths $= \frac{1}{n+1}\binom{2n}{n}$
$=$ nth Catalan number

But how do we prove our conjecture?

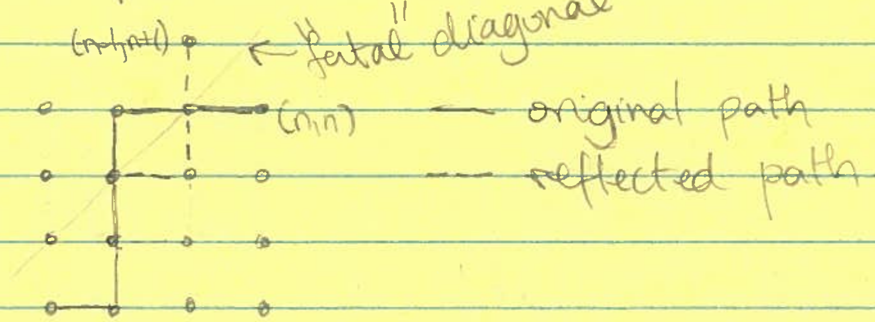
Pf Subtraction Principle.

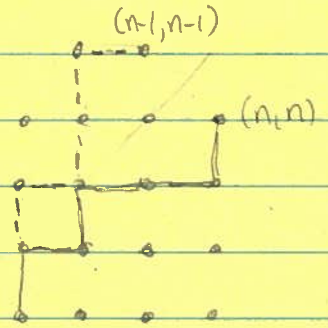
\# favorable paths = \# possible paths − \# unfavorable paths

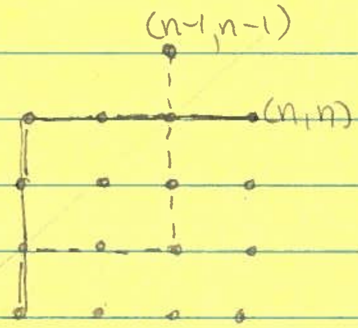
where an unfavorable path is one that crosses above the diagonal at least once.

At first glance, such paths would seem to be just as difficult to count as those that are confined to the lower half of the grid.

What makes them countable is an ingenious trick called the "reflection principle":







Evidently the reflection of an unfavorable path is a minimal length path [reflection just switches Es and Ns, without introducing Ws and Ss] from (0,0) to (n-1,n+1) [the image of (n,n) under reflection]

Conversely, a path from (0,0) to (n-1, n+1) must cross the fatal diagonal somewhere. After it first does this, reflect to obtain an unfavorable path.

Thus there is a bijection between minimal length paths from (0,0) to (n-1,n+1) and unfavorable paths.

Thus # unfavorable paths $= \binom{(n-1)+(n+1)}{n-1}$

$$= \binom{2n}{n-1}$$

Finally:

$$\#\text{favorable paths} = \binom{2n}{n} - \binom{2n}{n-1}$$

$$= \frac{(2n)!}{n!\,n!} - \frac{(2n)!}{(n-1)!\,(n+1)!}$$

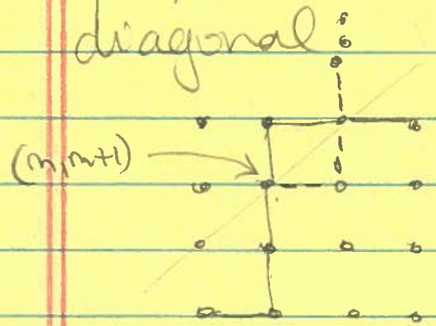
$$= \frac{(2n)!}{n!\,n!}\left[1 - \frac{n}{n+1}\right]$$

$$= \frac{(2n)!}{n!\,n!} \cdot \frac{1}{n+1}$$

$$= \frac{1}{n+1}\binom{2n}{n}$$

just as we hypothesized.

<u>Note</u> Let $(m, m+1)$ be the point at which an unfavorable path first touches the fatal diagonal:



	$\underbrace{mE,\ (m+1)N}$	$\underbrace{(n-m)E,\ (n-(m+1))N}$
—	ENNN	NEE
- - -	ENN	ENN

fatal diagonal

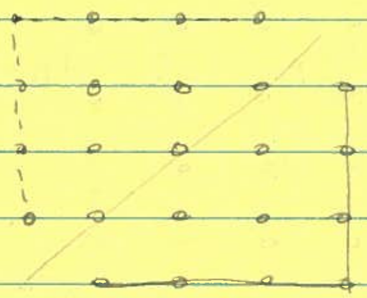
Since the reflected path switches Es and Ns "after" the fatal diagonal (relative to the original path), we can count its E's as follows:

$$\underbrace{m}_{\text{before diag.}} + \underbrace{n-(m+1)}_{\text{after diag.}} = n-1$$

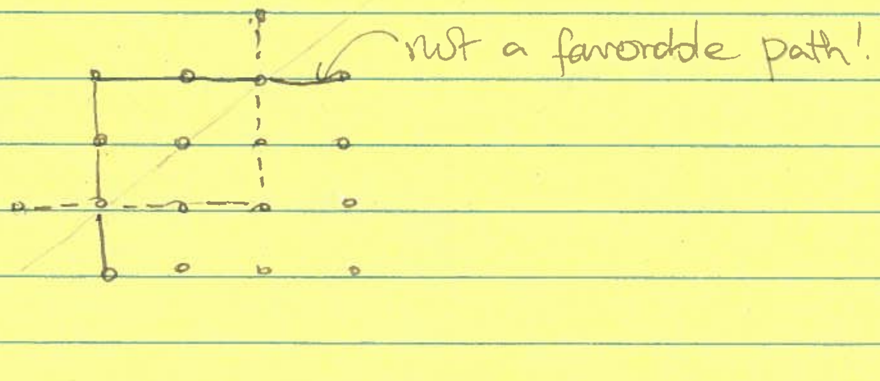
Similarly, #Ns on reflected path is

$$\underbrace{m+1}_{\text{before}} + \underbrace{n-m}_{\text{after}} = n+1$$

<u>Note</u>: The reflection principle does <u>not</u> establish a useful bijection for favorable paths, e.g.



While it is true that every favorable path, when reflected gives a minimal length path from $(-1,1)$ to $(n-1,n+1)$, it is not true that every minimal path from $(-1,1)$ to $(n-1,n+1)$ generates a favorable path:



Q. How many minimal length walks pass through (a,b)?

A. First walk from $(0,0)$ to (a,b): #ways $= \binom{a+b}{a}$

Then walk from (a,b) to (m,n): #ways $= \binom{m+n-(a+b)}{m-a}$

$$\#\text{walks thru } (a,b) = \binom{a+b}{a}\binom{m+n-(a+b)}{m-a}$$

$$\#\text{possible walks} = \binom{m+n}{m}$$

Thus: prob that random minimal length walk passes thru' (a,b) on an $m \times n$ grid is

$$\frac{\binom{a+b}{a}\binom{m+n-(a+b)}{m-a}}{\binom{m+n}{m}}$$

This is also the prob of getting "a" boys and 'm-a' girls in a random sample of m children taken from a collection of a+b boys and m+n-(a+b) girls.

This prob is usually written

$$P(\underset{\substack{\text{boys}\\\text{in}\\\text{sample}}}{k} ; \underset{\substack{\text{sample}\\\text{size}}}{r}, \underset{\text{boys}}{m}, \underset{\text{girls.}}{n}) = \frac{\binom{m}{k}\binom{n}{r-k}}{\binom{m+n}{r}}$$

and is called the hyper-geometric distribution