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L13 Aside: HW3 Q7c.

We flip a fair coin a number of times.

Show that getting $n-1$ or n heads in $2n$ flips is as likely as getting n or $n+1$ heads in $2n+1$ flips.

A.

# flips	# H
$2n$	$n-1$ or n
$2n+1$	n or $n+1$

$P(nH \text{ or } (n+1)H \text{ in } 2n+1 \text{ flips})$

$= P((n-1)H \text{ in } 2n \text{ flips}) \cdot \underbrace{P(H \text{ in next flip})}_{1/2}$

$+ P(nH \text{ in } 2n \text{ flips}) \circ \underbrace{P(H \text{ or } T \text{ in next flip})}_{1}$

$+ P((n+1)H \text{ in } 2n \text{ flips}) \cdot \underbrace{P(T \text{ in next flip})}_{1/2}$ (*)

Now:

$$P((n-1)H \text{ in } 2n \text{ flips}) = \frac{\binom{2n}{n-1}}{2^{2n}}$$

$$P((n+1)H \text{ in } 2n \text{ flips}) = \frac{\binom{2n}{n+1}}{2^{2n}}$$

$$\binom{2n}{n-1} = \binom{2n}{n+1} \quad \text{(symmetry of Pascals } \Delta\text{)}$$

Thus:

$$P((n-1)H \text{ in } 2n \text{ flips}) = P((n+1)H \text{ in } 2n \text{ flips})$$

Thus

(*) $= P((n-1)H \text{ in } 2n \text{ flips}) + P(nH \text{ in } 2n \text{ flips})$

$= P((n-1)H \text{ or } nH \text{ in } 2n \text{ flips})$

The Subtraction Principle / The Power of Negative Thinking

Q Flip a coin 5 times. How many ways can we get at least one H and at least one T?

A1 Enumerate:

# H	# T	# ways
0 ×	5 ✓	1
1 ✓	4 ✓	$\binom{5}{1}$
2 ✓	3 ✓	$\binom{5}{2}$
3 ✓	2 ✓	$\binom{5}{3}$
4 ✓	1 ✓	$\binom{5}{4}$
5 ✓	0 ×	1

favorable outcomes: $H = 1, 2, 3, 4$

Thus answer is $\binom{5}{1}+\binom{5}{2}+\binom{5}{3}+\binom{5}{4}$.

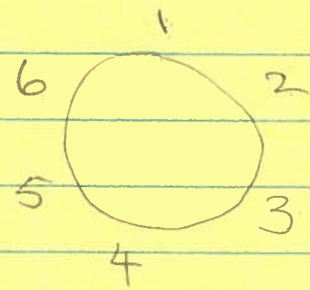
A2 Table highlights that it is easier to count the unfavorable outcomes, of which there are only two: all heads and all tails.

Since we know the # possible outcomes, 2^5, the answer can also be written:

$$2^5 - 2$$

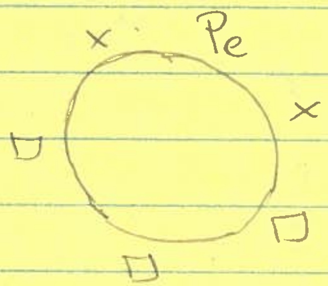
Q There are 6 people sitting around a table on labelled chairs. Two of them, Peter and Paul, are bitter enemies who refuse to sit together. How many seating arrangements are there?

A1



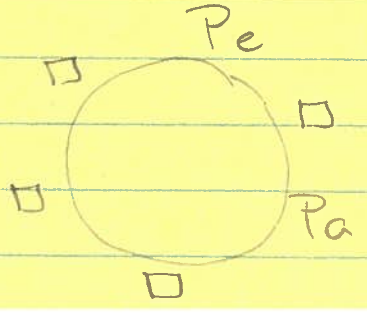
Note: if everyone just rotates one seat to the left, say, then this counts as a different arrangement.

1. Pick a seat for Peter. (6 possibilities)
2. Paul can sit in neither the seat to the left nor right of Peter:



$\Rightarrow$ Paul can sit in one of the 3 □'s.

3. Seat the remaining people:



$\Rightarrow$ 4 choices for 3rd person
3 " " 4th "
2 " " 5th "
1 " " 6th "

Thus, there are $6 \cdot 3 \cdot 4!$ ways to seat the people

A2 The "negative" approach to this question is to subtract # unfavorable arrangements from # possible arrangements.

An unfavorable arrangement is one where Peter and Paul are seated next to one another. In how many ways can this occur?

1. # pairs of adjacent seats = 6
2. # ways to seat Peter and Paul in chosen seats = 2
3. # ways to seat remaining people = 4!

$\Rightarrow$ # unfavorable arrangements = $6 \cdot 2 \cdot 4!$

possible arrangements = $6!$

$\Rightarrow$ # favorable arrangements $= 6! - 6 \cdot 2 \cdot 4!$

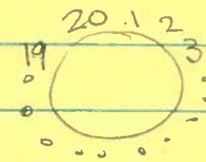
$$= 6(5-2)4!$$

$$= 6 \cdot 3 \cdot 4!$$

which is the same answer we got before.

Q There are 20 people seated at a circular table (with labeled chairs). How many ways can we pick a subset of 3 people containing no neighbors?

A1. Negative Approach.



Count #ways we can pick the people s.t. at least two of them are neighbors.

1. ways to pick two neighbors:

(1,2), (2,3), ... (18,19), (19,20), (20,1)

$\Rightarrow$ 20 ways.

2. #ways to pick 3rd member of set? You might think the answer is 18 since we have already chosen 2 of the 20 people. However this overcounts:

1 2 (3 4) 5 6 7 .. $\Rightarrow$ ..., {2,3,4}, {3,4,5}, ...
1 2 3 (4 5) 6 7 .. $\Rightarrow$ ..., {3,4,5}, {4,5,6}, ..
1 2 3 4 (5 6) 7 .. $\Rightarrow$ ..., {4,5,6}, {5,6,7}, ..

Evidently, the overcounting arises

when the 3^{rd} member is chosen to be a neighbor of the 1^{st} or 2^{nd} member.

Overcounting can be avoided by

(i) excluding one neighbor of the original pair from becoming the 3^{rd} member
$\Rightarrow$ # ways to choose 3^{rd} member $= 17$.
$\Rightarrow$ # 3-sets w/ at least 2 neighbors $= 20 \cdot 17$.

or

(ii) subtract 1 for each time a 3-set is double-counted, which is 20 (once for each of $(1,2), (2,3), \ldots, (19,20), (20,1)$).
Thus:

$$\#\text{3-sets w/ at least 2 neighbors} = 20 \cdot 18 - 20 = 20(18-1) = 20 \cdot 17.$$

There is another way to count the 3-sets w/ at least two neighbors. Partition the sets into those containing 2 and 3 neighbors.

1. 3-sets w/ 2 neighbors (only):

20 choices for the pair, and 16 choices for the remaining person:

	3-sets to exclude
$1 \; 2 \; (3 \; 4) \; \cancel{5} \; 6 \; 7 \; \ldots$	$\{2,3,4\}, \{3,4,5\}$
$1 \; 2 \; \cancel{3} \; (4 \; 5) \; \cancel{6} \; 7 \; \ldots$	$\{3,4,5\}, \{4,5,6\}$
$1 \; 2 \; 3 \; \cancel{4} \; (5 \; 6) \; \cancel{7} \; \ldots$	$\{4,5,6\}, \{5,6,7\}$
$\vdots$	$\vdots$

$\Rightarrow$ #ways $= 20 \cdot 16$

2. 3-sets w/ 3 neighbors: 20 choices:
$(1\,2\,3), (2\,3\,4), (3\,4\,5), \ldots, (19\,20\,1), (20,1,2)$

Then

$$\#\text{ 3-sets w/ at least 2 nbrs} = 20 \cdot 16 + 20 = 20 \cdot 17$$

as before.

Finally: #ways to pick 3 people s.t. <u>no</u> 2 of them are neighbors is:

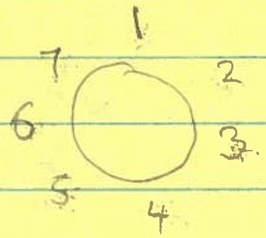
$$\underbrace{\binom{20}{3}}_{\substack{\text{\#ways to choose} \\ \text{3 people from 20.}}} - 20 \cdot 17$$

A2: Direct Approach"

Consider simplest example:

#ways to choose A : 7

#ways to choose B : 7-3=4 (exclude A and his nbrs)



(A,B,C)	#ways	shared nbr?
(1,3,5)	2	✓
(1,3,6)		
(1,4,6)	1	×
(1,5,3)	1	×
(1,6,3)	2	✓
(1,6,4)		

(2,4,6)	2	(3,5,7)	2
(2,4,7)		(3,5,1)	
(2,5,7)	1	(3,6,1)	1
(2,6,4)	1	(3,7,5)	1
(2,7,4)	2	(3,1,5)	2
(2,7,5)		(3,1,6)	

...

Pattern:

- shared nbr: $\ldots \underbrace{X\,A\,X\,B\,X}_{\text{not candidates for } C} \ldots$ ⇒ #ways $= 7-5$
- no shared nbr: $\ldots \underbrace{X\,A\,X\,X\,B\,X}_{6} \ldots$

  or $\ldots \underbrace{X\,A\,X}_{3} \ldots \underbrace{X\,B\,X}_{3} \ldots$

  either way: 6 cannot be candidates for C ⇒ #ways $= 7-6$

How many ways to choose B s.t. it shares a nbr with A?

$$\left.\begin{array}{l} \cdots XAXB \cdots \\ \cdots BXAX \cdots \end{array}\right\} \text{2 ways.}$$

How many ways to choose B s.t. it does not share a nbr with A?

$$\cdots \underbrace{XXAXX}_{\substack{\text{none of these} \\ \text{can be B}}} \cdots \implies \#\text{ways} = 7 - 5 = 2$$

Return to original problem involving 20 people. #ways to choose a triple (A, B, C) s.t. no two elements are nbrs is:

$$\underbrace{20}_{\text{choose A}} \cdot \Big\{ \underbrace{2}_{\substack{\text{choose} \\ \text{B s.t. it} \\ \text{shares nbr} \\ \text{with A}}} \cdot \underbrace{(20-5)}_{\text{choose C}} + \underbrace{(20-5)}_{\substack{\text{choose} \\ \text{B s.t. it} \\ \text{doesn't} \\ \text{share nbr} \\ \text{with A.}}} \underbrace{(20-6)}_{\text{choose C.}} \Big\}.$$

$$= 20(2 \cdot 15 + 15 \cdot 14) = 20 \cdot 15 \cdot 16$$

Finally, observe that the question asked for a set of 3 people, i.e. the order in which the people are picked is irrelevant. We can "forget" this order by dividing by 3!

Thus

ways to choose 3 people s.t. no 2 are nbrs

$$= \frac{20.15.16}{3!}$$

This is the same as the original answer, $\binom{20}{3} - 20\,17$, since

$$\frac{20.15.16}{3!} = 20 \cdot 5 \cdot 8 = 20 \cdot 40$$

and:

$$\binom{20}{3} - 20.17 = \frac{20.19.18}{6} - 20.17$$

$$= 20\,[19 \cdot 3 - 17]$$

$$= 20\,[57 - 17]$$

$$= 20 \cdot 40$$

$$\begin{array}{r} 19 \\ \times\ 3 \\ \hline 57 \end{array}$$