412 Counting: labeled vs unlabeled.

Q. How many ways can we make 2 teams of 5 out of a class of 10 students?

A1. Note: once we pick the members of one team (red), the other team (blue) is "picked automatically". Thus, there are $\binom{10}{5}$ ways to make the 2 teams. Or are there?

It turns out that artificially labeling the teams creates an error. For example, suppose the students are $\{1,2,\ldots,10\}$. Then $\binom{10}{5}$ counts

Red	Blue
$\{1,2,3,4,5\}$	$\{6,7,8,9,10\}$

and

$\{6,7,8,9,10\}$	$\{1,2,3,4,5\}$

as distinct ways to make a team. In other words, our labeling methodology has double-counted the # ways to make 2 teams.

Thus correct answer is $\frac{1}{2}\binom{10}{5}$.

A2 There is a clever way to avoid the double-counting issue. We choose one of the students Peter, and count the number of ways of choosing his team mates. That means choosing 4 teammates from the remaining 9 students, which can be done in $\binom{9}{4}$ ways. This is also # ways of making the teams since Peter is on one, and only one, of the 2 teams.

This is the correct way to label the teams ("Peter's team", "not Peter's team") to avoid double counting.

Q. How many ways can we make 2 teams of 4 and 6 out of the class?

A1 The teams are now effectively labeled; eg smaller team and bigger team.

#ways to build the smaller team is $\binom{10}{4}$, which is also #ways to build the larger team, since $\binom{10}{4} = \binom{10}{10-4} = \binom{10}{6}$

small team $\{1, 2, 3, 4\}$ big team $\{5, 6, 7, 8, 9, 10\}$.

You see that I can't write:

small team $\{5, 6, 7, 8, 9, 10\}$ big team $\{1, 2, 3, 4\}$.

which is what led to the double-counting issue when dividing into teams of equal size.

A2 Choose particular student, Peter.

Consider arrangements where Peter is on the small team. #ways to choose his teammates is $\binom{9}{3}$. = #ways to choose his opponents, $\binom{9}{6}$.

Alternatively, Peter is on the big team.
#ways to choose his teammates = $\binom{9}{5}$
= #ways to choose his opponents = $\binom{9}{4}$.
Since Peter is on the small or large team, and not both, total # ways is:

$$\binom{9}{3} + \binom{9}{4} = \binom{10}{4}$$

by:

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \binom{n}{k}$$

Q Peter and Paul are two students in the class. What is the probability that they are on the same team? [Team size = 5]

A1. If they are on the same team, then there are 3 remaining slots on the team to be filled with the remaining $10-2=8$ students.

ways to do this is $\binom{8}{3}$

total # ways to build 2 teams is $\binom{9}{4}$.

Thus: prob $= \binom{8}{3} \Big/ \binom{9}{4}$, assuming all possible team arrangements are equally likely.

A2. We can also get the answer using "labelled" teams, say "red" and "blue".

ways that Peter and Paul could be on red team $= \binom{8}{3}$ $\{\underbrace{\text{Peter, Paul}, \circ, \circ, \circ}_{\text{red}}\}$.

ways that they could be on blue team $= \binom{8}{3}$ $\{\underbrace{\text{Peter, Paul}, \circ, \circ, \circ}_{\text{blue}}\}$.

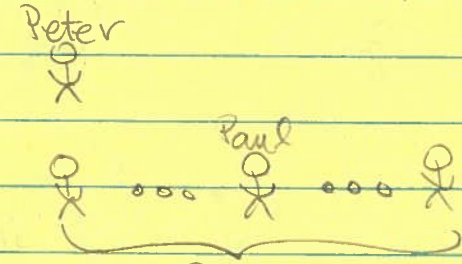
total #ways to build two teams, when the teams are labelled / colored is: $\binom{10}{5}$

eg $\underbrace{\{1,2,3,4,5\}}_{\text{red}}$ distinct from $\underbrace{\{1,2,3,4,5\}}_{\text{blue}}$

Thus:

$$\text{prob} = \frac{\binom{8}{3} + \binom{8}{3}}{\binom{10}{5}} = \frac{2\binom{8}{3}}{\binom{10}{5}} = \frac{\binom{8}{3}}{\frac{1}{2}\binom{10}{5}} = \frac{\binom{8}{3}}{\binom{9}{4}}$$

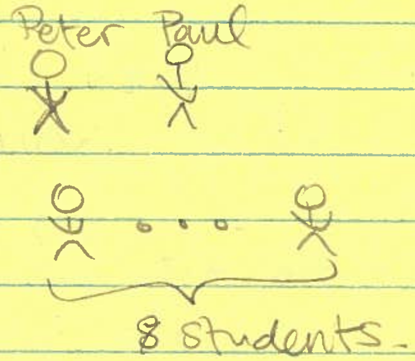
A3 Imagine generating a team containing Peter and Paul by picking students one-by-one. We first pick Peter and move him to one side [we always pick Peter first]:



9 students

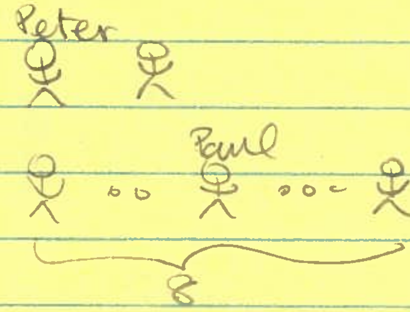
We next pick one student from the 9 "uniformly at random", ie. each of the 9 are equally likely to be chosen.

Prob that we choose Paul is 1/9, in which case we have a favorable outcome, ie. a team containing Peter and Paul:



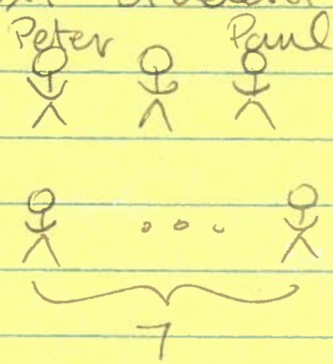
$$P(\text{Pe Pa} \ldots) = \frac{1}{9}$$

But What if we hadn't chosen Paul?:



$$\text{Prob} = \frac{8}{9}$$

Well, we could still generate a team containing Peter and Paul if, say, the next student to be picked were Paul:



But what is the probability of this event?

For this to happen, two things must occur, in order: (choose non-Paul) AND

(choose Paul)
Prob of former event is $\frac{8}{9}$.
The prob of the latter event, conditional upon the first occurring, is $\frac{1}{8}$.

Multiplying probs (since both events must occur), we get:

$$P(Pe \circ Pa \circ \circ) = \frac{8}{9} \cdot \frac{1}{8} = \frac{1}{9}.$$

We can repeat this argument to conclude that:

$$P(Pe \circ \circ Pa \circ) = \frac{8}{9}\frac{7}{8}\frac{1}{7} = \frac{1}{9}$$

$$P(Pe \circ \circ \circ Pa) = \frac{8}{9} \cdot \frac{7}{8}\frac{6}{7}\frac{1}{6} = \frac{1}{9}$$

Finally: net prob. that Peter and Paul are on same team is:

$$P(PePa \circ \circ \circ) + P(Pe \circ Pa \circ \circ) + P(Pe \circ \circ Pa \circ) + P(Pe \circ \circ \circ Pa)$$

$$= \frac{4}{9}$$

which can be written as $\binom{8}{3} / \binom{9}{4}$, since the 5! factors in the numerator and denominator cancel.

MAJOR NOTE: Each team, e.g. $\{Pe, Pa, O_1, O_2, O_3\}$, can be picked in $4!$ ways, assuming I always pick Peter first, eg.

$$(Pe, Pa, O_1, O_2, O_3)$$
$$(Pe, Pa, O_1, O_3, O_2)$$
$$\vdots$$
$$(Pe, O_1, Pa, O_2, O_3)$$
$$(Pe, O_1, Pa, O_3, O_2)$$
$$\vdots$$

etc.

After Peter I need to pick $\{Pa, O_1, O_2, O_3\}$ and there are $4!$ orders in which I would pick them.

The fact that a single team (set) corresponds to many picking orders (permutations) might suggest that I am overcounting. I am not. Why? Because I'm not counting teams, I am counting ways to generate teams!

(Much) Simpler probabilistic calculation

Peter sits on one of the two teams:

Pe _ _ _ _ | _ _ _ _ _

The remaining 9 students occupy the 9 unfilled slots above

Now focus on Paul. There are 9 slots where Paul could sit, but only 4 of them is "favorable", e.g.:

favorable outcome: Pe _ Pa _ _ | _ _ _ _ _

unfavorable outcome: Pe _ _ _ _ | _ Pa _ _ _

favorable outcomes = 4.

possible outcomes = 9

$\Rightarrow$ prob that Peter and Paul sit on same team = $4/9$.

Some Probability Theory

I want to take this opportunity to formalize the calculation above, which should help for the probability stuff to come later in the course.

Let X_i be a Boolean random variable. This means it takes only two values:

$$X_i = \begin{cases} 1 & \text{if } i^{th} \text{ person picked is Paul} \\ 0 & \text{" " " " not Paul} \end{cases}$$

Then

$$P(\text{Pe Pa} \circ\circ\circ) = P(X_2 = 1)$$

$$= P(\{X_2 = 1\})$$

where

$\{X_2 = 1\}$ = set of all outcomes where 2^{nd} person picked is Paul.

$$= \{(Pe, Pa, O_1, O_2, O_3), \\ (Pe, Pa, O_1, O_2, O_4), \\ (Pe, Pa, O_1, O_2, O_5), \ldots\}.$$

But what is $P(A)$ where A is a set of outcomes.

The answer depends upon what we consider to be the set of all possible outcomes, Ω, because then

$$P(A) = \frac{|A|}{|\Omega|} \quad ; A \subset \Omega .$$

In our case, .

$$\Omega = \{ (Pe, Pa, O_1, O_2, O_3), \\ (Pe, Pa, O_1, O_2, O_4), \ldots \\ (Pe, O_1, Pa, O_2, O_3), \ldots \\ (Pe, O_1, O_2, Pa, O_3), \ldots \}.$$

$$= \{X_2 = 1\} \cup \{X_2 = 0\}.$$

$$= \{(Pe\, Pa \ldots)\} \cup \{(Pe\, O_1 \ldots)\} \cup \{(Pe\, O_2 \ldots)\} \cup \ldots$$

Since these 9 subsets are disjoint and of equal size, say $|\{X_2 = 1\}|$, we have:

$$P(X_2 = 1) = \frac{|\{X_2 = 1\}|}{|\Omega|} = \frac{|\{X_2 = 1\}|}{9\,|\{X_2 = 1\}|} = \frac{1}{9}.$$

What about our other calculation:

$$P(X_2=0, X_3=1) = P(Pe, O_i, Pa, O_j, O_k)$$

An axiom of prob says:

$$P(A \cap B) = \underbrace{P(A|B)}_{\text{"prob of A given B"}} P(B)$$

In our case:

$$P(X_2=0, X_3=1) = P(X_3=1, X_2=0)$$

$$= P(X_3=1 \mid X_2=0)\, P(X_2=0).$$

Now:

$$P(X_2=0) = 1 - P(X_1=0) = 1 - \tfrac{1}{9} = 8/9.$$

and

$$P(X_3=1 \mid X_2=0) = \frac{1}{8}$$, as argued before

Thus:

$$P(X_2=0, X_3=1) = \frac{1}{8}\cdot\frac{8}{9} = \frac{1}{9}$$, as before.

Here's another way to compute $P(X_2=0, X_3=1)$:

$$|\{X_2=0, X_3=1\}| = |\{(Pe, \_, Pa, \_, \_)\}|$$

$$= \binom{8}{3} \cdot 3! = \frac{8!}{5!}$$

$$|\Omega| = |\{(Pe, \_, \_, \_, \_)\}|$$

$$= \binom{9}{4} \cdot 4! = \frac{9!}{5!}$$

Thus

$$P(X_2=0, X_3=1) = \frac{8!/5!}{9!/5!} = \frac{1}{9}$$