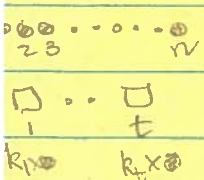
L11 **Multinomial Coefficients & Multinomial Theorem**

Last time we defined the multinomial coefficient. If $k_1 + \cdots + k_t = n$, then

$$\binom{n}{k_1, k_2, \ldots, k_t} = \begin{cases} \dfrac{n!}{k_1! \, k_2! \cdots k_t!} & \text{each } k_i \geq 0. \\ 0 & \text{otherwise} \end{cases}$$

Here is another way to think about multinomial coefficients. Suppose we have n objects and t boxes labelled $1, 2, \ldots, t$. Then the multinomial coefficient above is the number of ways to place k_1 of the objects in box 1, k_2 in box 2, etc., without regard to the order of the objects in each box.



Making the multinomial coefficient zero when at least one k_i is negative makes sense since it is impossible to place a negative number of objects in a box.

Suppose $t=3$:

arrangement	# ways
$\boxed{1}$ k_1 ● $\quad \boxed{2}$ k_2 ● $\quad \boxed{3}$ k_3 ●	$\binom{n}{k_1, k_2, k_3}$

Now relabel box 2 and box 3:

$$\boxed{1} \quad \boxed{3} \quad \boxed{2}$$
$$k_1 \bullet \quad k_2 \bullet \quad k_3 \bullet$$

and reorder boxes:

arrangement

$$\boxed{1} \quad \boxed{2} \quad \boxed{3}$$
$$k_1 \bullet \quad k_3 \bullet \quad k_2 \bullet$$

ways

$$\binom{n}{k_1\, k_3\, k_2}$$

Since every arrangement before relabeling maps to an arrangement after relabeling and reordering, we have:

$$\binom{n}{k_1\, k_2\, k_3} = \binom{n}{k_1\, k_3\, k_2}$$

In general, we have:

Suppose $\pi(1), \dots, \pi(t)$ is a permutation of $\{1, \dots, t\}$, then:

$$\binom{n}{k_1, \dots, k_t} = \binom{n}{k_{\pi(1)}, \dots, k_{\pi(t)}}$$

Recall Pascals identity:

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k} \quad (*) \quad (\text{cf } L6)$$

This is a special case of a more general identity for multinomial coefficients. For example, when $t=2$, the general identity is:

$$\binom{n}{k_1, k_2} = \binom{n-1}{k_1-1, k_2} + \binom{n-1}{k_1, k_2-1} \quad ; \; k_1+k_2=n$$

which is just $(*)$. If $t=3$, the general identity is:

$$\binom{n}{k_1, k_2, k_3} = \binom{n-1}{k_1-1, k_2, k_3} + \binom{n-1}{k_1, k_2-1, k_3} + \binom{n-1}{k_1, k_2, k_3-1} \quad (**)$$

To prove the $(t=3)$ case, we can generalize the counting proof of $(*)$ (presented in L6):

Choose an object α.
It must be placed in one of 3 boxes
Suppose we place it in box 1.
How many such arrangements?
$n-1$ objects must be placed in 3 boxes s.t. k_1-1 sit in box 1, k_2 sit in box 2, k_3 in box 3:

$$\#\text{ways} = \binom{n-1}{k_1-1, k_2, k_3}$$

Suppose instead that we place α in box 2. How many ways?
$n-1$ objects must be placed in 3 boxes s.t. k_1 in box 1, k_2-1 in box 2, k_3 in box 3:

$$\# \text{ways} = \binom{n-1}{k_1, k_2-1, k_3}$$

Similar argument if we place α in box 3.

Since α must be placed in one, and only one, of the 3 boxes, we have (**).

$\square$

Just like binomial coefficients may be arranged in a triangle, trinomial coefficients ($t=3$) may be arranged in a pyramid.

First, let's review how we built Pascals triangle

First choose a row, indexed by $n = 0, 1, 2, \ldots$. Then position the binomial coeffs $\binom{n}{k}$, $k = 0, 1, \ldots, n$, according to the value of k:

$$\binom{n}{0}\binom{n}{1}\cdots\binom{n}{n}$$

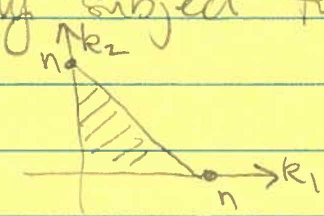
Pascals triangle can be split into rows, because once n is chosen, there is only one "degree of freedom", k. This is so because:

$$\binom{n}{k} = \binom{n}{k, n-k}$$
$$= \binom{n}{k_1, k_2} \quad \text{where } k_2 = n - k_1$$

When $t=3$, lines (rows) are replaced by planes because there are now 2 degrees of freedom for each n:

$$\binom{n}{k_1, k_2, k_3} \quad \text{with } k_3 = n - k_1 - k_2 = n - (k_1 + k_2)$$

can vary these independently subject to $k_1 + k_2 \leq n$ and $k_1, k_2 \geq 0$:

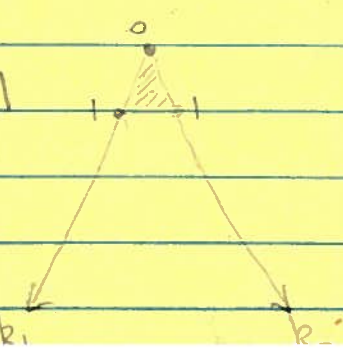


We can rearrange the axes above so that the allowed (k_1, k_2) tuples form an equilateral triangle. For example:

$n=0$ $\Leftrightarrow$ $\binom{0}{0,0,0}$

k_1 k_2

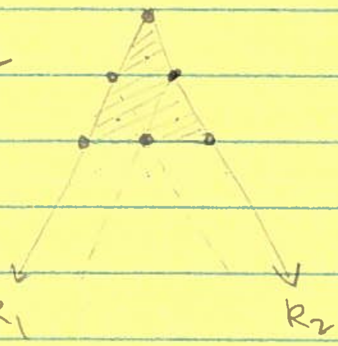
$n=1$



$\Leftrightarrow$ $\binom{1}{0,0,1}$

$\binom{1}{1,0,0}$ $\binom{1}{0,1,0}$

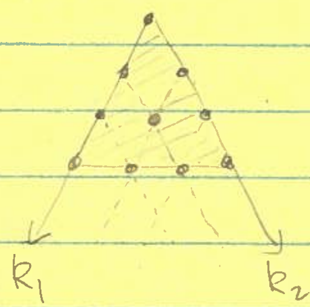
$n=2$



$\Longleftrightarrow$

$$\binom{2}{0,0,2}$$
$$\binom{2}{1,0,1} \quad \binom{2}{0,1,1}$$
$$\binom{2}{2,0,0} \quad \binom{2}{1,1,0} \quad \binom{2}{0,2,0}$$

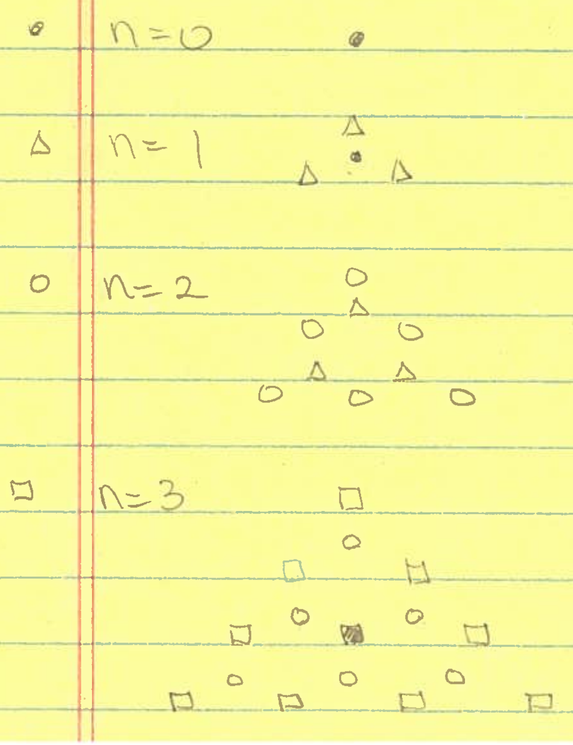
$n=3$



$\Longleftrightarrow$

$$\binom{3}{0,0,3}$$
$$\binom{3}{1,0,2} \quad \binom{3}{0,1,2}$$
$$\binom{3}{2,0,1} \quad \binom{3}{1,1,1} \quad \binom{3}{0,2,1}$$
$$\binom{3}{3,0,0} \quad \binom{3}{2,1,0} \quad \binom{3}{1,2,0} \quad \binom{3}{0,3,0}$$

Having constructed the triangles above we next stack them to form a pyramid:



Notice that the ■ in the (n=3) layer is surrounded by a triangle of o:

$$\binom{2}{1,0,1} \circ \qquad \circ \binom{2}{0,1,1}$$
$$\blacksquare$$
$$\binom{3}{1,1,1}$$
$$\circ$$
$$\binom{2}{1,1,0}$$

We see that taking 1 off

each of the k_i's in the $(n=3)$ coefficient.
yields the nearest coefficients in the $(n=2)$ layer.

Thus we see that the identity $(**)$ - the generalization of Pascals identity - provides an easy way to construct Pascals pyramid layer-by-layer.

The addition identity for multinomial coefficients, eg $(**)$ for $t=3$, can be used to prove the multinomial theorem:

$$(x_1 + \cdots + x_t)^n = \sum_{k_1 + \cdots + k_t = n} \binom{n}{k_1, k_2, \ldots, k_t} x_1^{k_1} \cdots x_t^{k_t}$$

Note: When $t=2$ this reduces to the binomial theorem.

You'll prove the theorem in HW4 by generalizing the proof for the binomial case.