L10
§3.1 LVP

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

Ex

$$(x+y)^0 = 1$$

$$(x+y)^1 = 1\cdot y + 1\cdot x$$

$$(x+y)^2 = 1\cdot y^2 + 2\cdot xy + 1\cdot x^2$$

$$(x+y)^3 = 1\cdot y^3 + 3\cdot xy^2 + 3\cdot x^2 y + 1\cdot x^3$$

Pf #1 Induction:

True for $n=0$

Suppose true for n.

Consider

$$(x+y)^{n+1} = (x+y)(x+y)^n$$

$$= (x+y)\cdot \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

$$= \underbrace{\sum_{k=0}^{n} \binom{n}{k} x^{k+1} y^{n-k}}_{\sum_{k=1}^{n+1} \binom{n}{k-1} x^k y^{n-k+1}} + \underbrace{\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k+1}}_{\binom{n}{0} x^0 y^{n+1} + \sum_{k=1}^{n} \text{—"—}}$$

$$= \sum_{k=1}^{n} \text{—"—} + \binom{n}{n} x^{n+1} y^0$$

Thus

$$(x+y)^{n+1} = \sum_{k=1}^{n} \left[\binom{n}{k-1} + \binom{n}{k}\right] x^k y^{(n+1)-k} + \binom{n}{n} x^{n+1} y^0 + \binom{n}{0} x^0 y^{n+1}$$

$$= \sum_{k=1}^{n} \binom{n+1}{k} x^{k} y^{(n+1)-k} + \binom{n+1}{n+1} x^{n+1} y^{0} + \binom{n+1}{0} x^{0} y^{n+1}$$

$$= \sum_{k=0}^{n+1} \binom{n+1}{k} x^{k} y^{(n+1)-k}$$

$\boxtimes$

Consider

$$(x+y)^3 = (x+y)(x+y)\cdot(x+y)$$

How do we get the final expansion:

$$x^3 + 3x^2y + 3xy^2 + y^3 \ ?$$

Start w/ the highest power of x: x^3. How many ways to do it:

$$(x+y)(x+y)(x+y)$$

$\Rightarrow$ just one way

How about xy^2?

$$(x+y)(x+y)(x+y) \longrightarrow xy^2$$

$$(x+y)(x+y)(x+y) \longrightarrow xyx$$

$$(x+y)(x+y)(x+y) \longrightarrow yx^2$$

$\Rightarrow$ 3 ways

Do you see the pattern? Each term of the expansion is $\alpha x^k y^{3-k}$, where α is the # ways of choosing k x's from $\{x, x, x\}$, where order doesn't matter.

Pf #2 This argument can be applied in general to conclude that

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k}$$

☒

Special cases of the Binomial Theorem

$x=1, y=1$ $\quad 2^n = \sum_{k=0}^{n} \binom{n}{k}$

$x=2, y=1$ $\quad 3^n = \sum_{k=0}^{n} \binom{n}{k} 2^k$

⋮

$x=-1, y=1$ $\quad 0 = \sum_{k=0}^{n} \binom{n}{k} (-1)^k$ "alternating coeffs" in a row of Pascals triangle

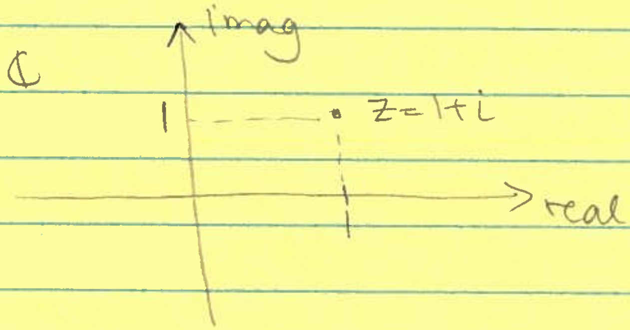
But what happens if we only alternate even terms in a row?

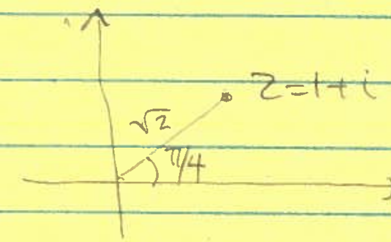
To answer that question, we need

complex numbers...

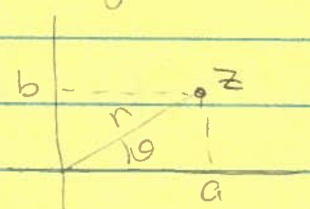
Quick Review of Complex Numbers

$\mathbb{C}$





In general: $z = a + bi = re^{i\theta} = r(\cos\theta + i\sin\theta)$



$\Rightarrow a = r\cos\theta;\ b = r\sin\theta$

and $|z| = r = \sqrt{a^2+b^2}$

Multiplying complex numbers: $i^2 = -1$

$\Rightarrow (a+bi)(c+di) = ac - bd + (bc+ad)i$.

$r_1 e^{i\theta_1} \cdot r_2 e^{i\theta_2} = r_1 r_2 e^{i(\theta_1+\theta_2)}$

$x = i,\ y = 1$ $\quad (1+i)^n = \sum_{k=0}^{n} \binom{n}{k} i^k$

But:

$$LHS = \left(\sqrt{2}\, e^{i\pi/4}\right)^n = 2^{n/2} e^{in\pi/4}$$

$$= 2^{n/2}\cos\left(\frac{n\pi}{4}\right) + i\, 2^{n/2}\sin\left(\frac{n\pi}{4}\right).$$

Now break up RHS into real & imag parts by examining i^k:

k	0	1	2	3	4	...
i^k	1	i	-1	$-i$	1	...

$$\underbrace{1 \quad i \quad -1 \quad -i}_{\text{cycle.}} \quad \underbrace{1 \quad \cdots}_{\text{cycle.}}$$

Clearly, for even k, i^k alternates between $+1$ and -1, while for odd k, i^k alternates between i and $-i$.

Equating real(LHS) and real(RHS) yields:

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{2k} (-1)^k = 2^{n/2} \cos\left(\frac{n\pi}{4}\right)$$

and equating imag(LHS) and imag(RHS) $\Rightarrow$

$$\sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n}{2k+1} (-1)^k = 2^{n/2} \sin\left(\frac{n\pi}{4}\right)$$

When n is even:

$$\sum_{k=0}^{n} \binom{2n}{2k} (-1)^k = 2^n \cos\left(\frac{n\pi}{2}\right)$$

$$\sum_{k=0}^{n-1} \binom{2n}{2k+1} (-1)^k = 2^n \sin\left(\frac{n\pi}{2}\right).$$

Here is another special case of the Binomial Thm:

$x = x, y = 1$ $\qquad (1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k$

Take derivative of both sides w.r.t. x:

$$n(1+x)^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k x^{k-1}$$

Plug in $x=1$:

$$n\, 2^{n-1} = \sum_{k=0}^{n} \binom{n}{k} k$$

Recall Vandermonde identity:

$$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k} \binom{n}{r-k}$$

Binomial Pf

We have

$$\sum_{r=0}^{m+n} \binom{m+n}{r} x^r \underset{\text{binomial thm}}{=} (1+x)^{m+n} = (1+x)^m (1+x)^n$$

$$\underset{\text{binomial thm}}{=} \sum_{\alpha=0}^{m} \binom{m}{\alpha} x^\alpha \cdot \sum_{\beta=0}^{n} \binom{n}{\beta} x^\beta \qquad (*)$$

But:

$$\sum_{\alpha} a_\alpha x^\alpha \sum_{\beta} b_\beta x^\beta$$

$$= (a_0 x^0 + a_1 x^1 + \cdots)(b_0 x^0 + b_1 x^1 + \cdots)$$

Consider the x^2 term in the expansion:

$$a_0 x^0 b_2 x^2 + a_1 x^1 \cdot b_1 x^1 + a_2 x^2 \cdot b_0 x^0$$

$$= \left[\sum_{k=0}^{2} a_k b_{2-k}\right] x^2 .$$

In general, the x^r term in the expansion is:

$$\left[\sum_{k=0}^{r} a_k b_{r-k}\right] x^r$$

which in the case of (*) is:

$$\binom{m+n}{r} x^r = \left[\sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k}\right] x^r$$

ie. $$\binom{m+n}{r} = \sum_{k=0}^{r} \binom{m}{k}\binom{n}{r-k} .$$

$\blacksquare$

§2.3 HHM **Multinomial Coefficents**

Q. How many anagrams of BANANAS?

A. There are $7!$ permutations of the letters, but many of them yield the same word since we can't distinguish the 3 A's or the 2 N's: Thus there are

$$\frac{7!}{3! \cdot 2!}$$

distinct anagrams.

Alternative answer: Imagine constructing an anagram:

$$\_\ \_\ \_\ \_\ \_\ \_\ \_$$

First place the A's: $\binom{7}{3}$ ways, eg

$$\underline{A_1}\ \_\ \underline{A_2}\ \_\ \_\ \underline{A_3}\ \_ \ = \ \underline{A_3}\ \_\ \underline{A_2}\ \_\ \_\ \underline{A_1}\ \_$$

Then place the N's: $\binom{4}{2}$ ways, eg

$$\underline{A}\ \underline{N_1}\ \underline{A}\ \underline{N_2}\ \_\ \underline{A}\ \_ \ = \ \underline{A}\ \underline{N_2}\ \underline{A}\ \underline{N_1}\ \_\ \underline{A}\ \_$$

Then place B: $\binom{2}{1}$ ways

Then place S: $\binom{1}{1}$ ways.

Total # ways = $\binom{7}{3}\binom{4}{2}\binom{2}{1}\binom{1}{1}$

This leads us to a proposition:

Prop: The # ways of dividing n objects into groups of size $k_1, k_2, \ldots, k_t$ where $k_1 + k_2 + \cdots + k_T = n$ is given by

$$\frac{n!}{k_1! \cdot k_2! \cdots k_T!} = \binom{n}{k_1}\binom{n-k_1}{k_2}\binom{n-k_1-k_2}{k_3}\cdots\binom{n-k_1-k_2\cdots-k_{t-1}}{k_t}$$

[Expand RHS to see how it reduces to LHS!]

$\frac{n!}{k_1!\cdots k_t!}$ is # ways of painting n labelled balls with t colors where k_i are of color i, and $k_1 + \cdots + k_t = n$.
To see this, think of the slots in the BANANAS example as the balls, the distinct letters {A, N, B, S} as the colors and $k_1 = 3$, $k_2 = 2$, $k_3 = 1$, $k_4 = 1$.

Def^n Multinomial coefficient is $n!/k_1!\,k_2!\cdots k_t!$ and denoted

$$\binom{n}{k_1, k_2, \ldots, k_t}$$